Directed Graphs of Entanglement Two

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Wrocław, 04.09.09

Graph Complexity Measures

- Complexity Measure:
  complexity : Graphs $\longrightarrow$ Natural Numbers
- Graph complex $\Rightarrow$ complexity(Graph) large
- On graphs with bounded measure: difficult problems become easy.

Example

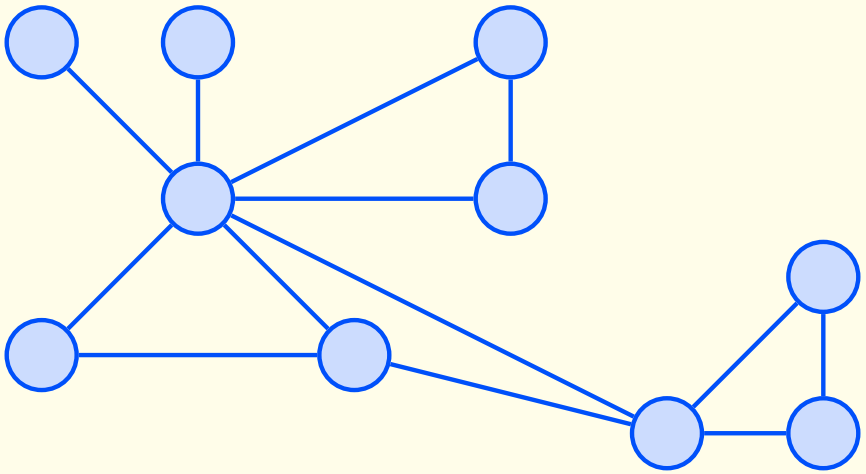
Hamiltonian Cycle, 3-Colorability, Vertex Cover

- on undirected graphs
- if tree-width is bounded

Tree-width

3 cops:

robber:



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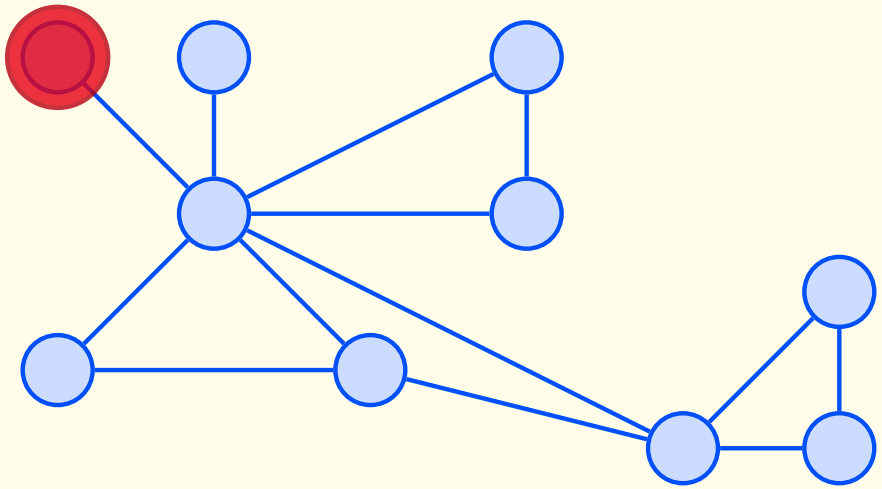

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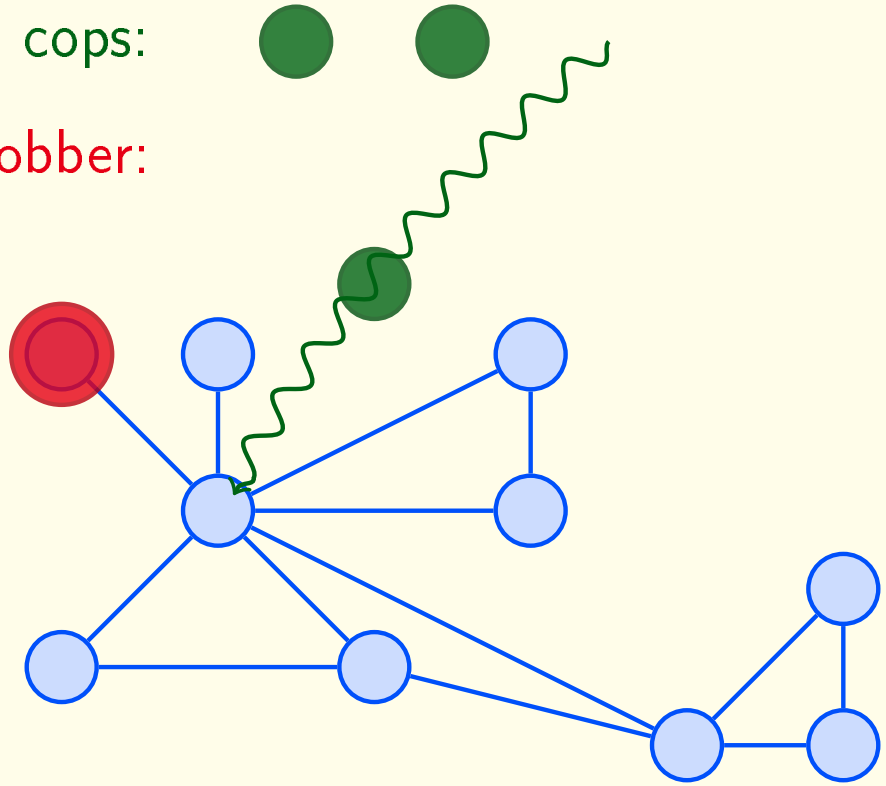

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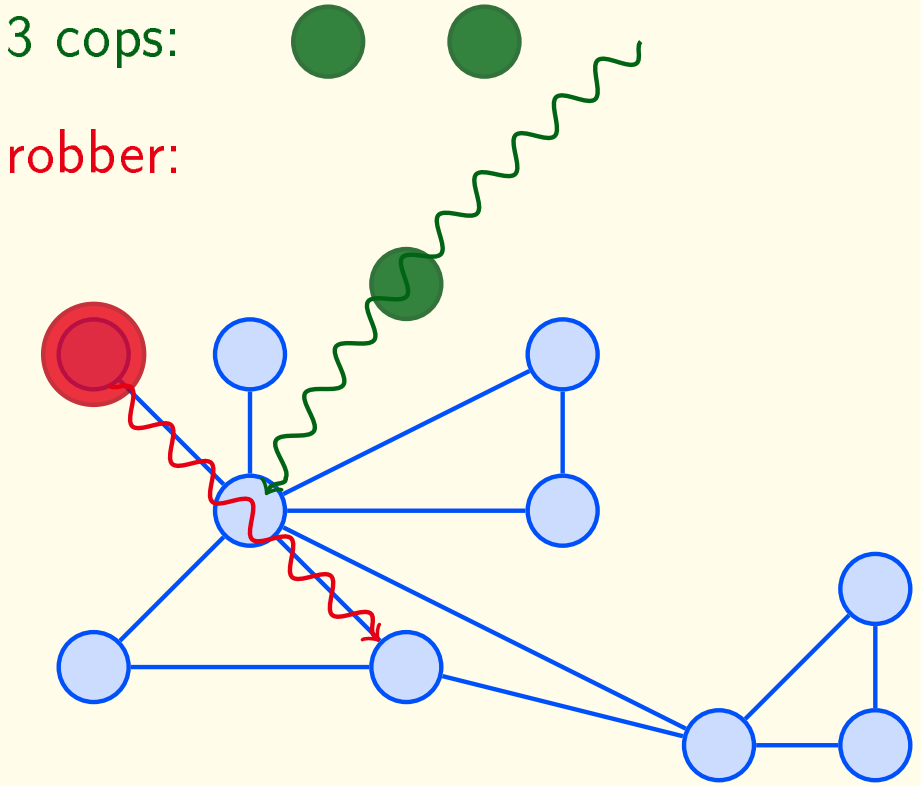

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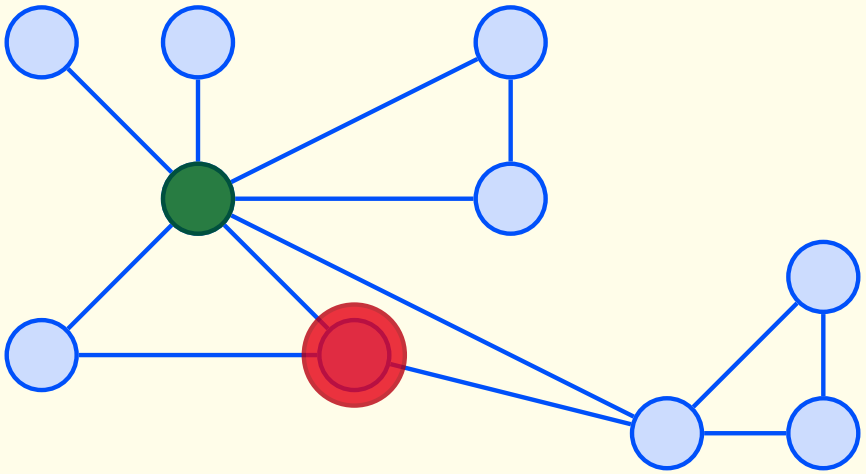

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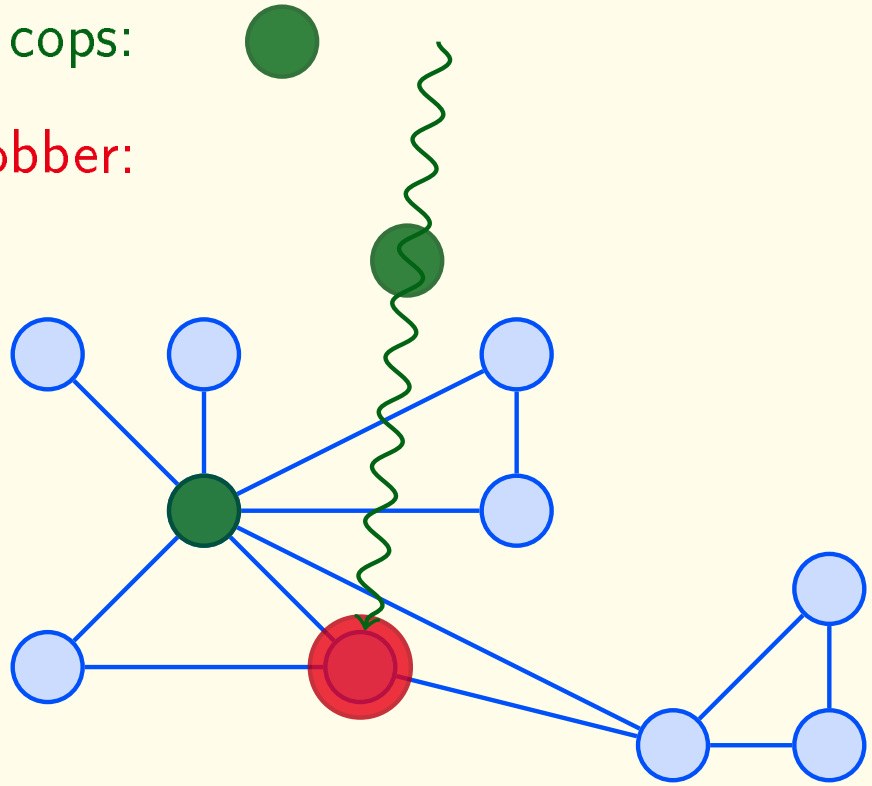

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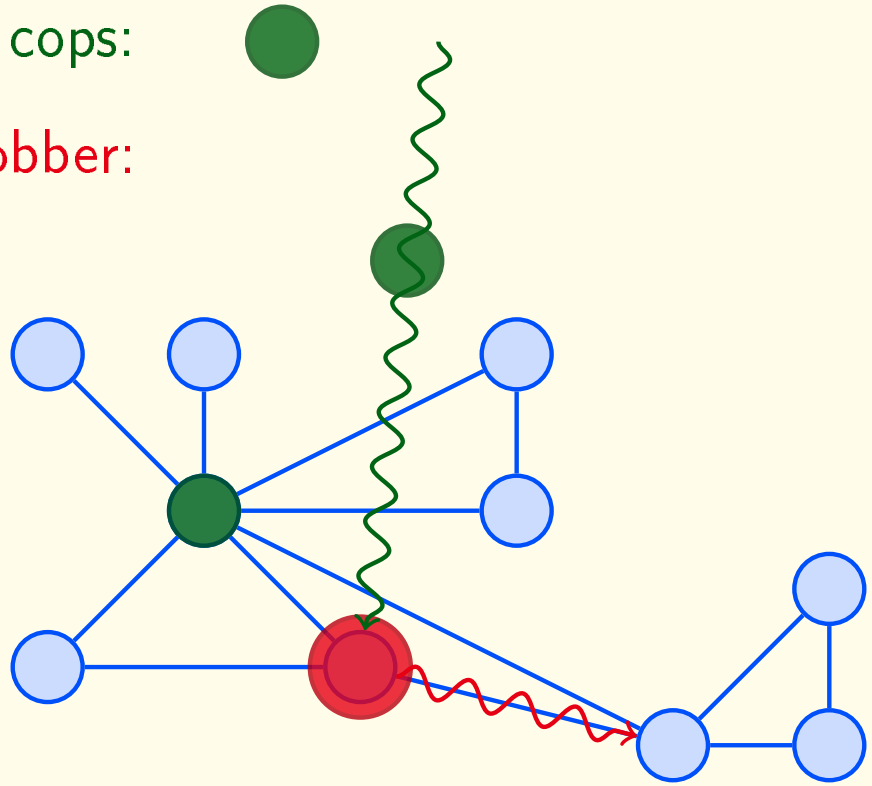

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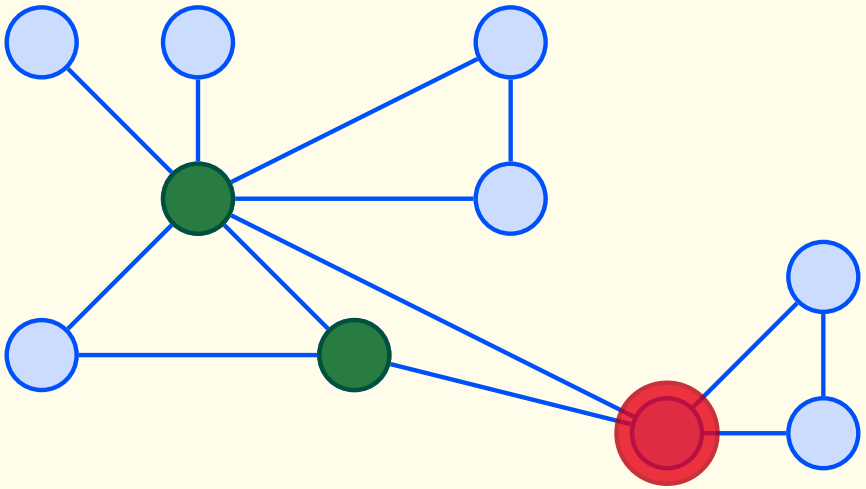

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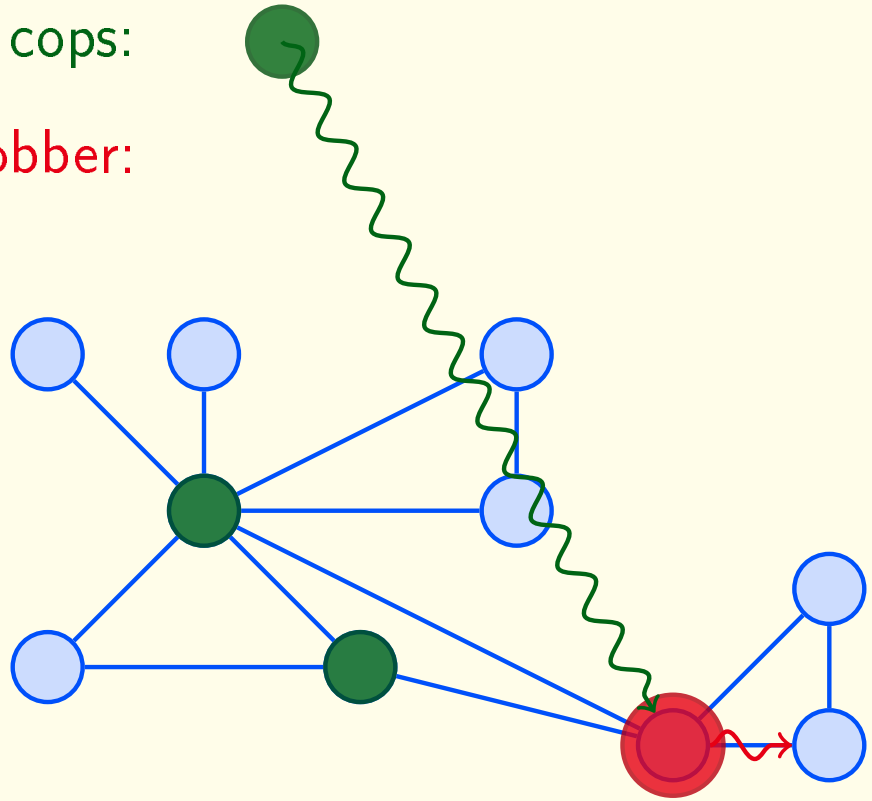

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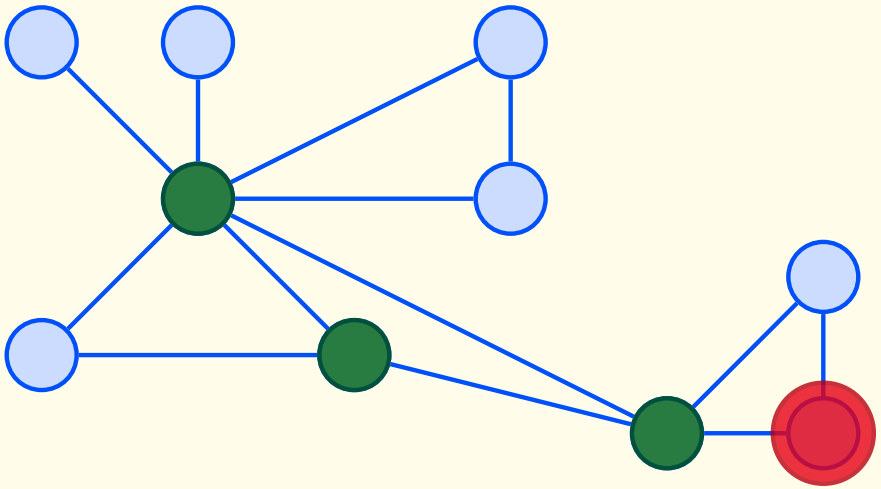

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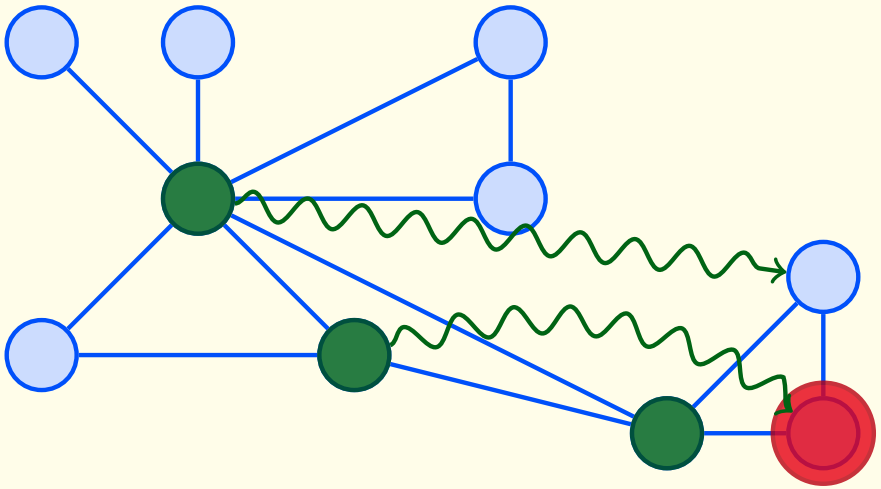

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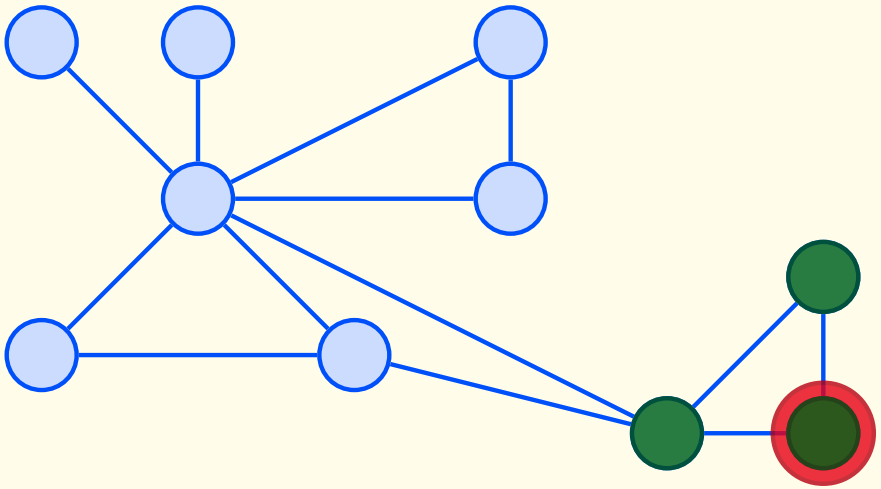
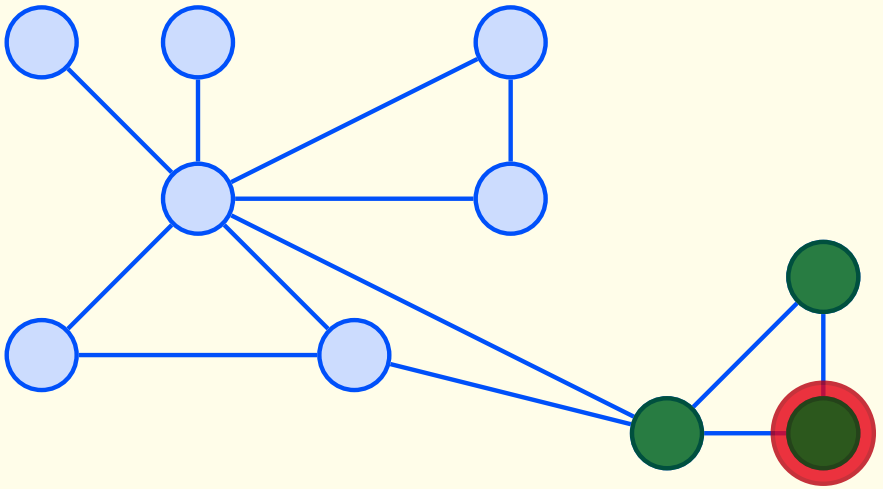


Tree-width

3 cops:

robber:

Tree-width -1 = minimum number of cops needed to capture the robber



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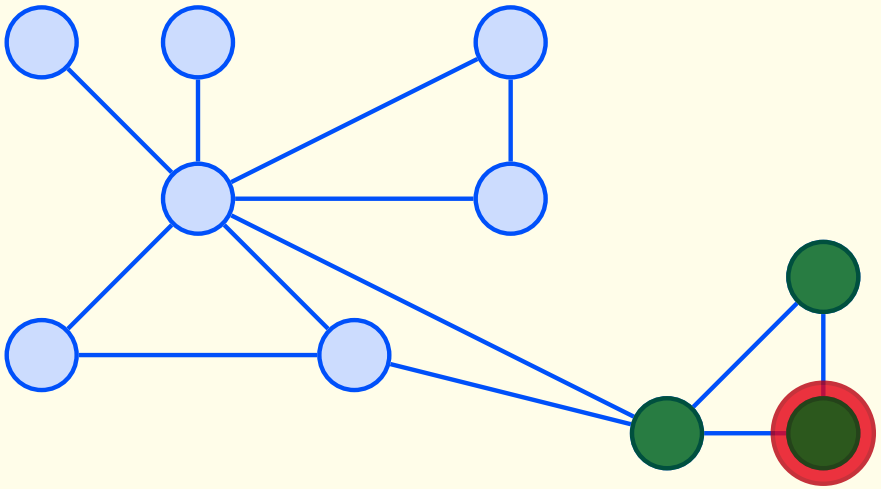

Why is it interesting?

Tree-width

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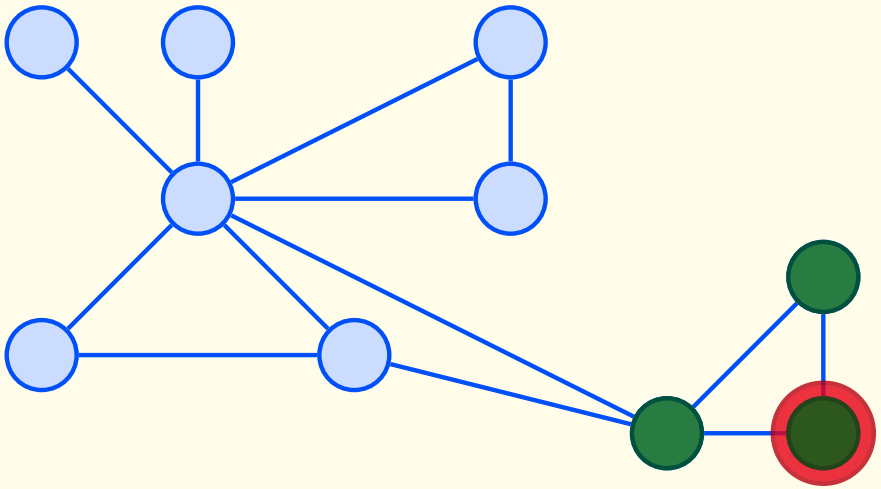
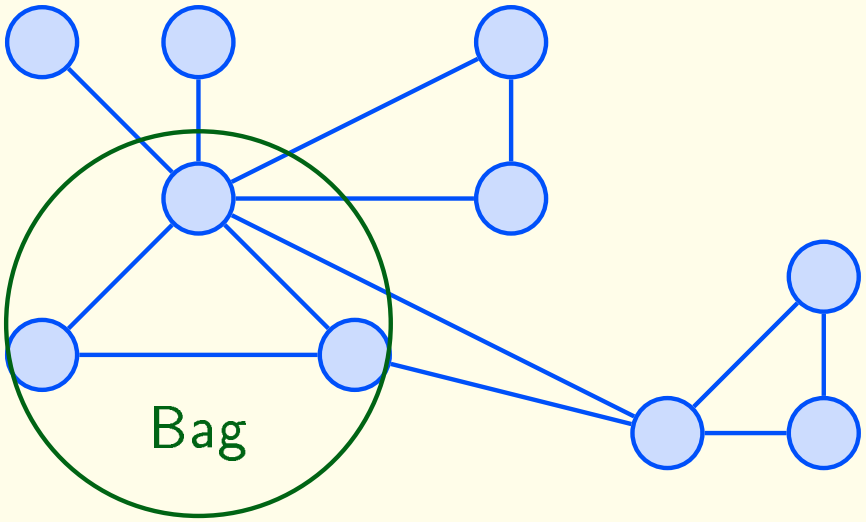


Decomposition!

Tree-width



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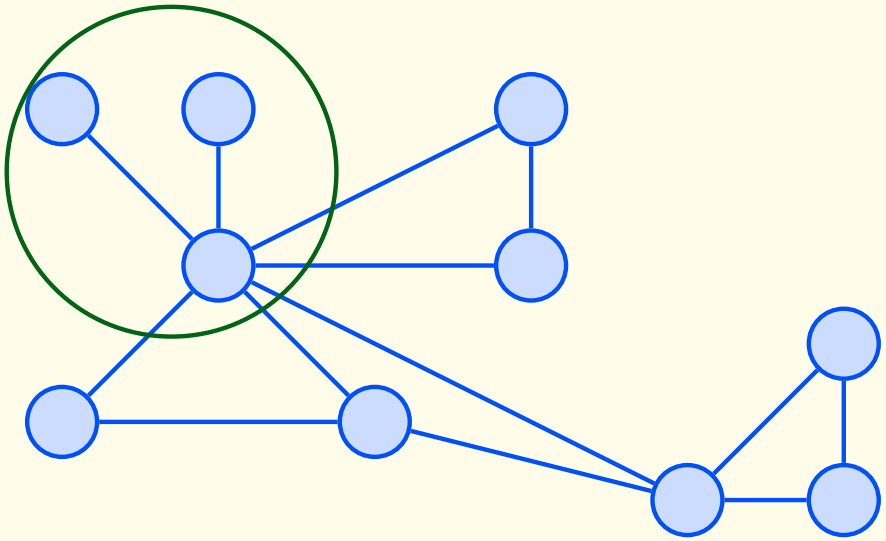

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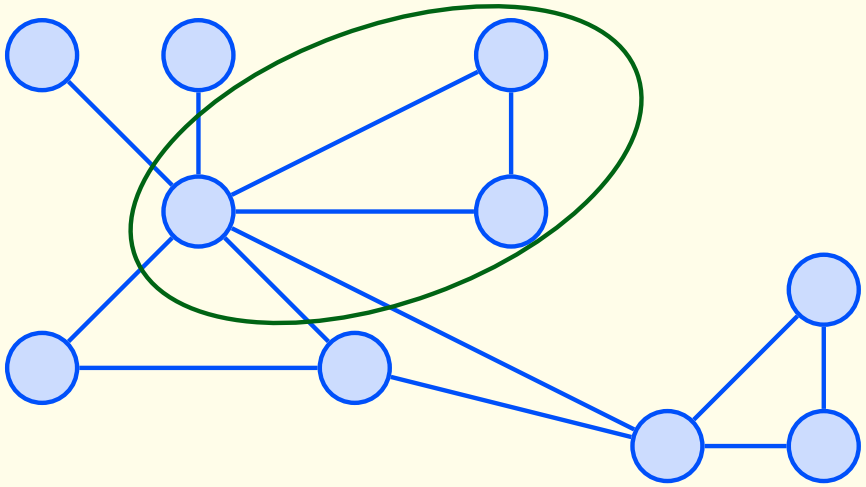

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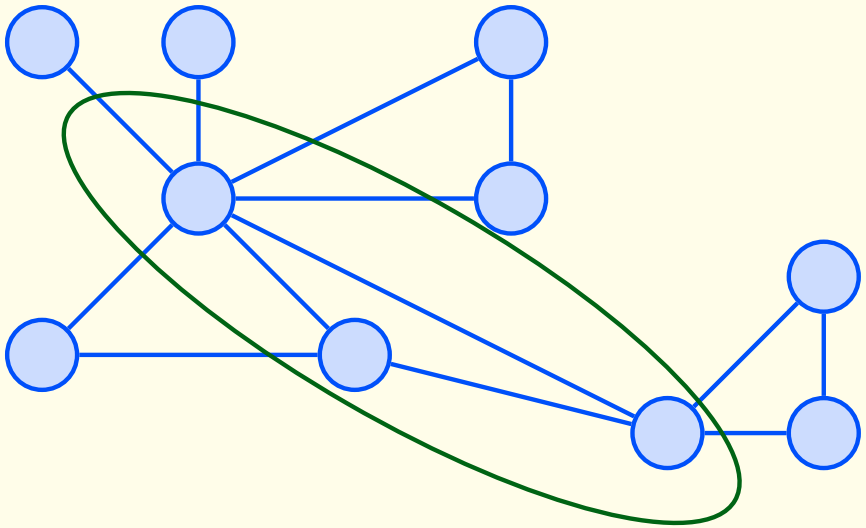

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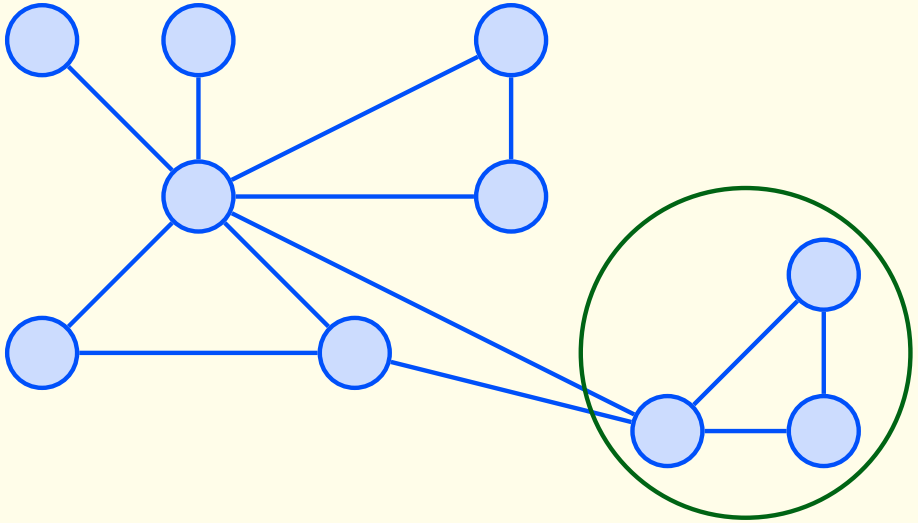

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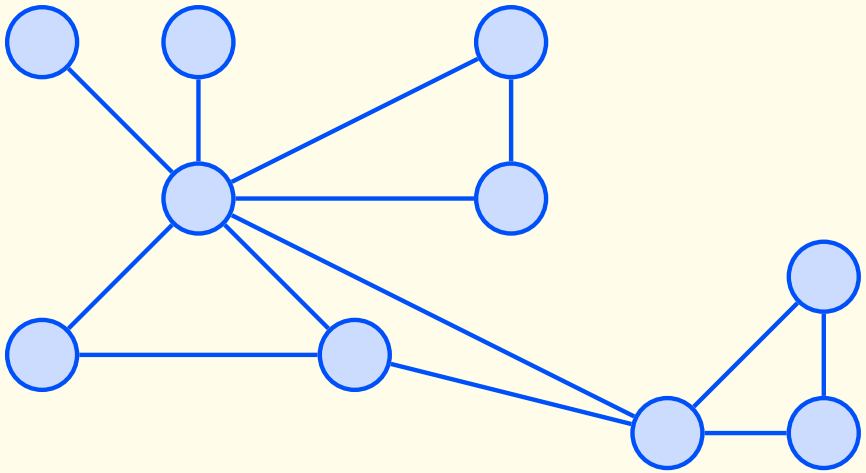

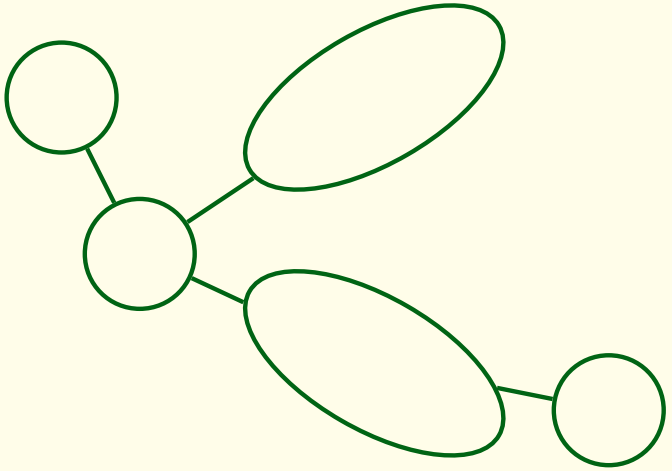

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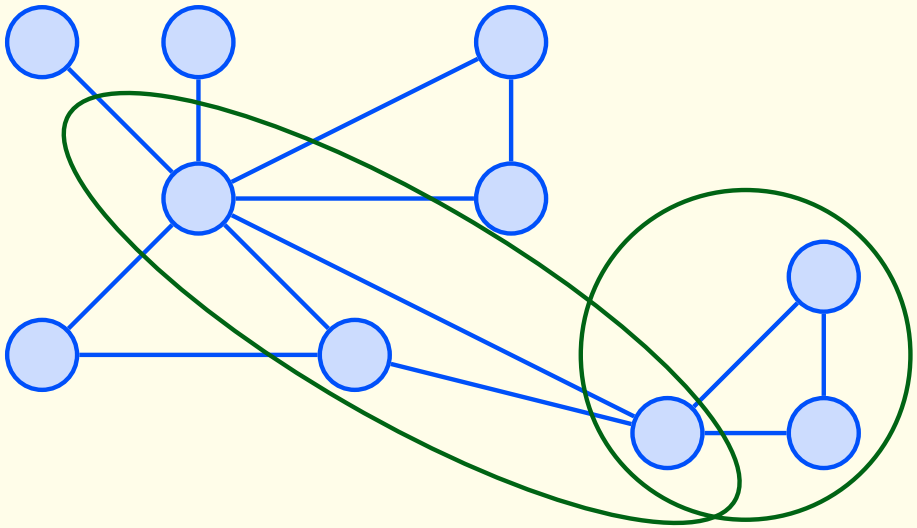
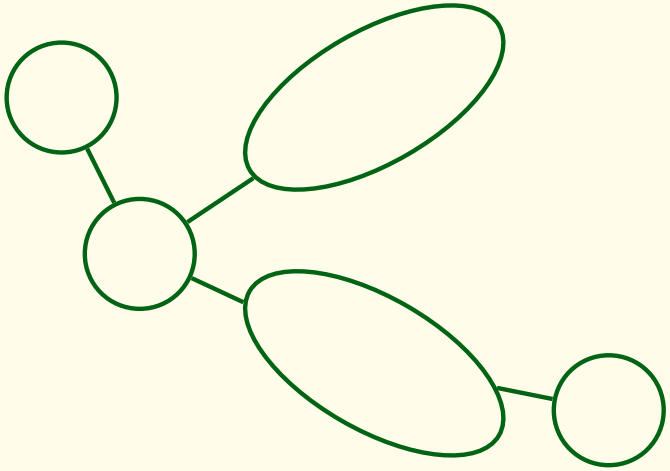
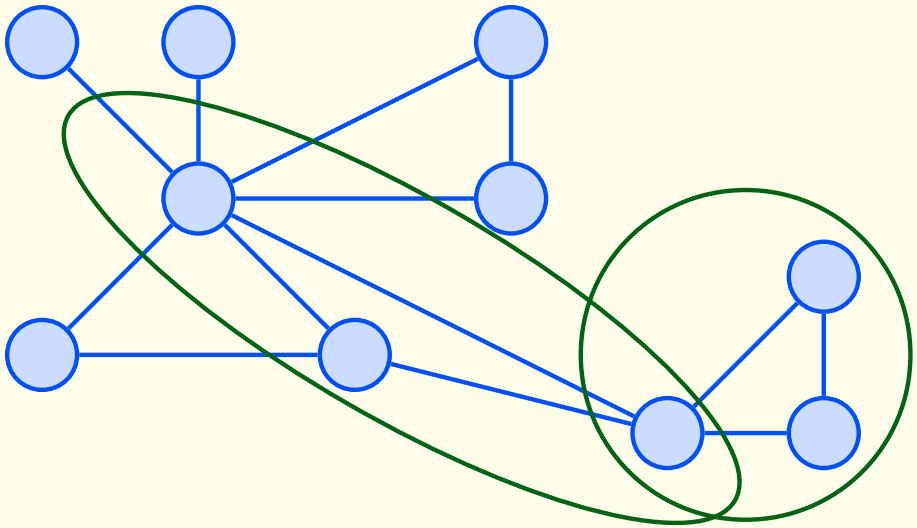
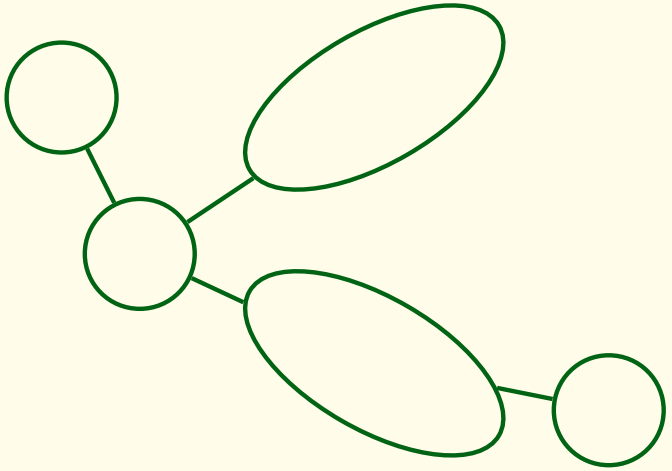




- Minimal possible bag size = tree-width -1.
- Bounded tree-width $\Rightarrow$ fast algorithms for difficult problems.

Tree-width





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Complexity Measures for Directed Graphs

- directed tree-width
- DAG-width
- Kelly-width

For all these measures

- characterisation by decomposition,
- fast algorithms, if measure bounded

Why Entanglement?

- Crucial element in proof: variable hierarchy of μ-calculus is strict (Berwanger, Grädel, Lenzi).
- Parity games are easy on graphs of bounded entanglement (Berwanger, Grädel).
- Winning regions of parity games definable in LFP on graphs of bounded entanglement (Dawar, Grädel).

- No corresponding decomposition $\Rightarrow$ no basis for fast algorithms!

Graphs of Entanglement Two

1. Characterisations:
   - a game characterisation
   - a decomposition
   - an inductive construction
2. DAG-width, Kelly-width
   - entanglement $= 2 \Rightarrow$ DAG-width, Kelly-width ≤ 3
   - There are graphs of DAG-width, Kelly-width 3 and unbounded entanglement (Berwanger, Dawar, Hunter, Kreutzer).
3. still interesting graphs (e.g., control-flow graphs of structured programs)
4. $\Rightarrow$ a basic class of graphs
5. A direct generalisation to more than two cops fails.

Entanglement: Rules of the Game

- Positions:
  - Robber occupies a vertex.
  - Cops are on vertices and outside graph.
  - Initial position: all players outside graph.
- Moves:
  - Robber goes along an edge (must move!) to a cop free vertex.
  - One cop can jump on robber's vertex or cops do nothing.
- Winning condition:
  - Cops win if robber cannot move.
- Entanglement = minimal number of cops needed to capture the robber.

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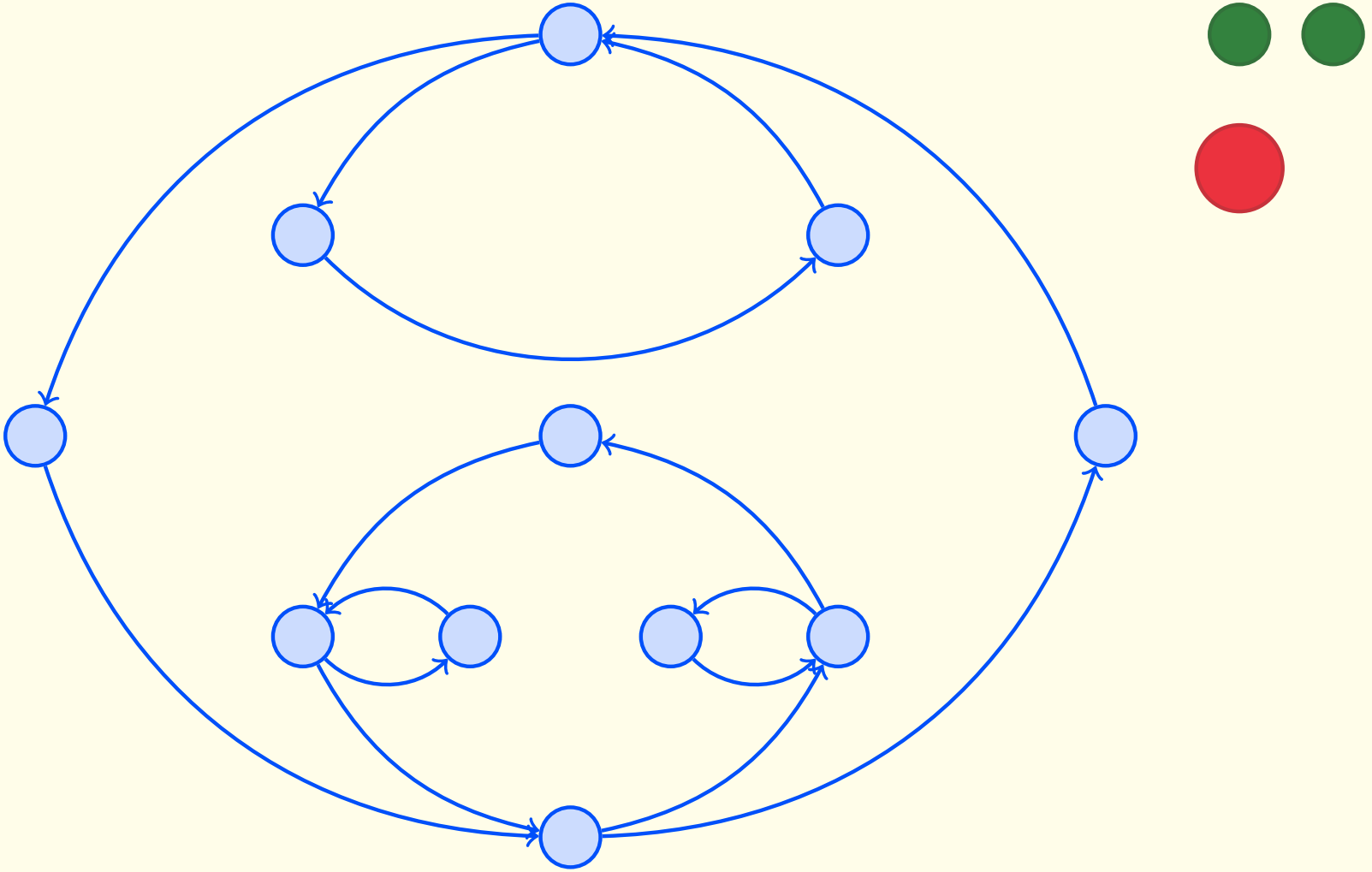

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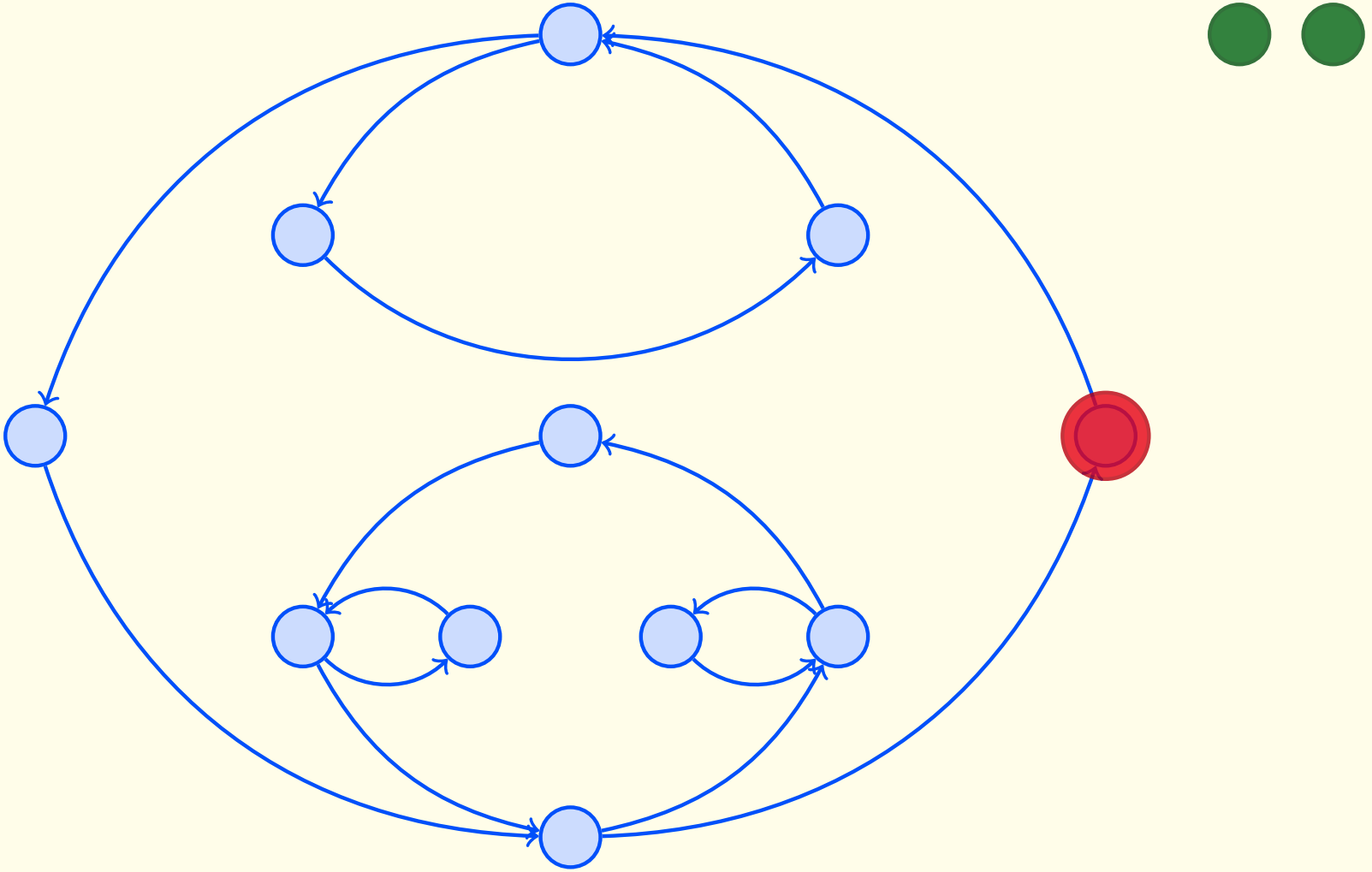

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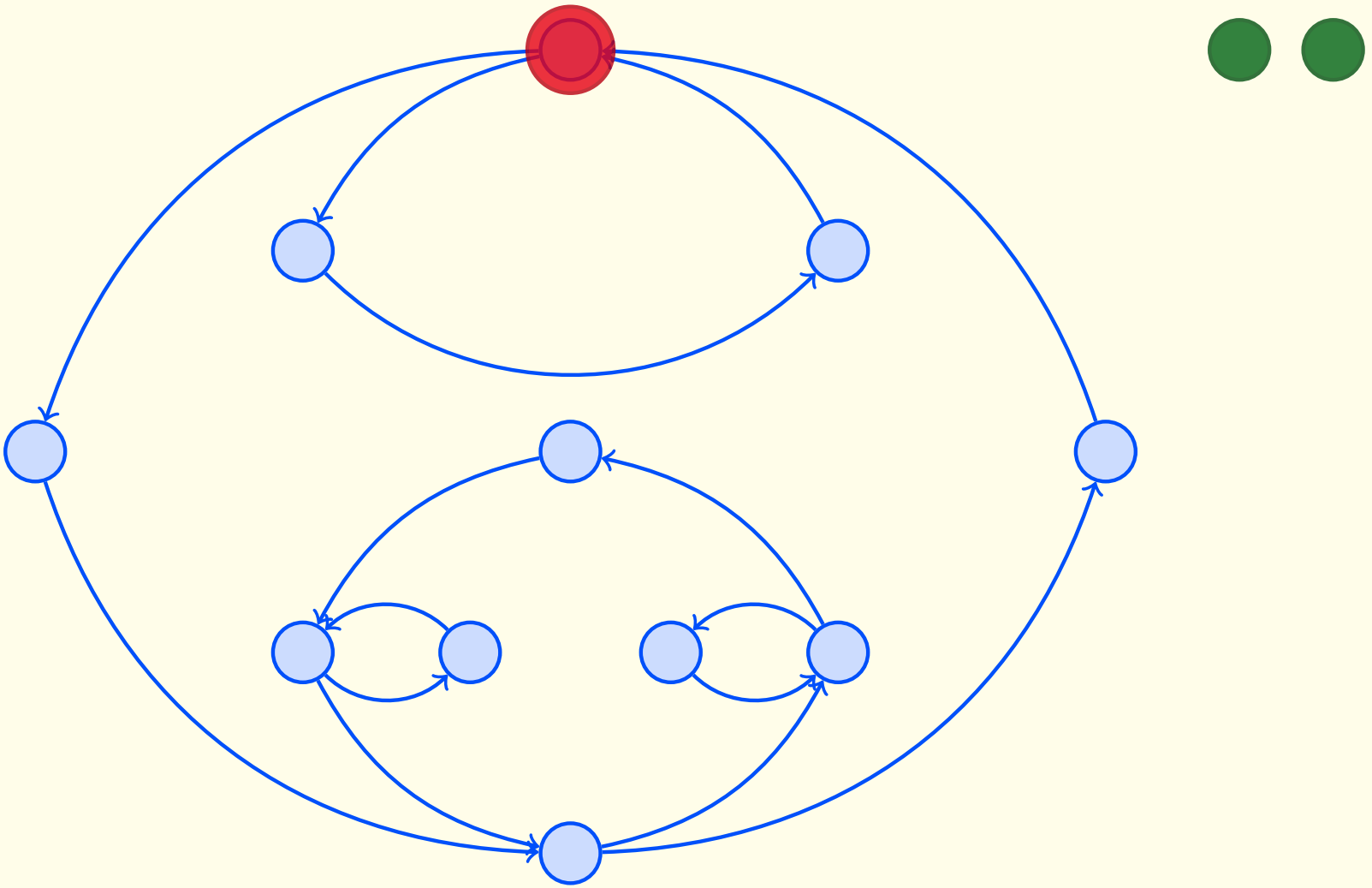

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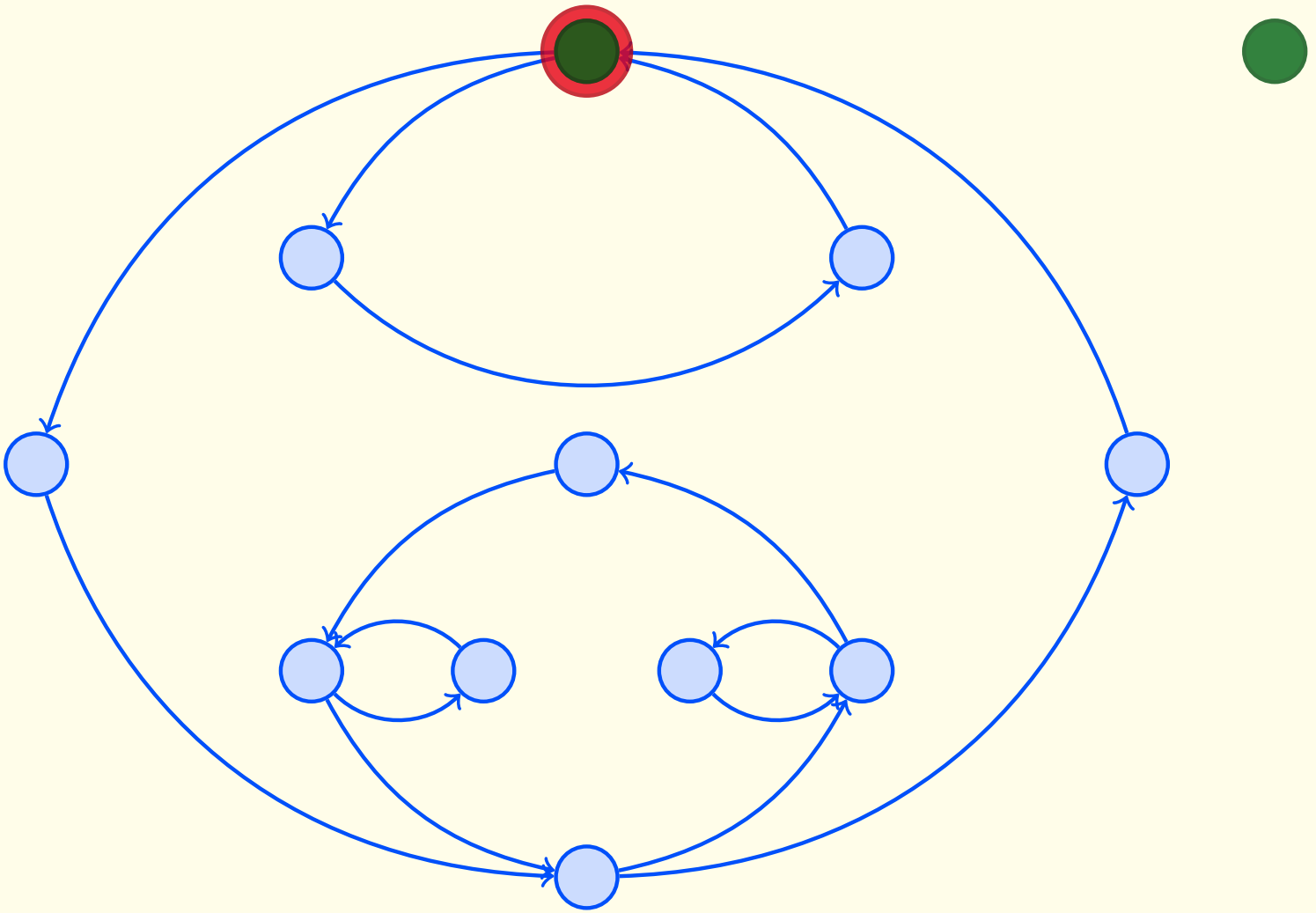

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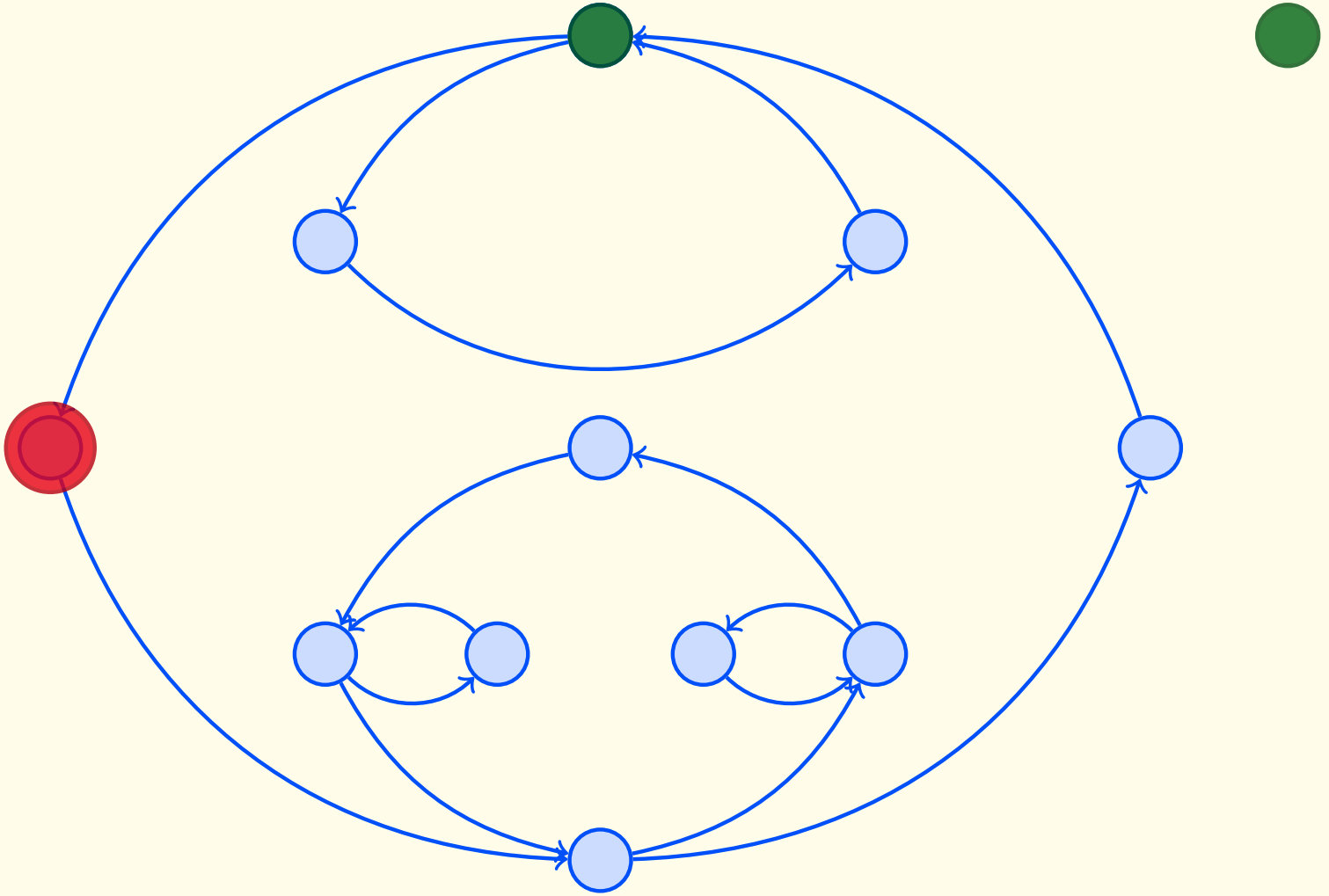

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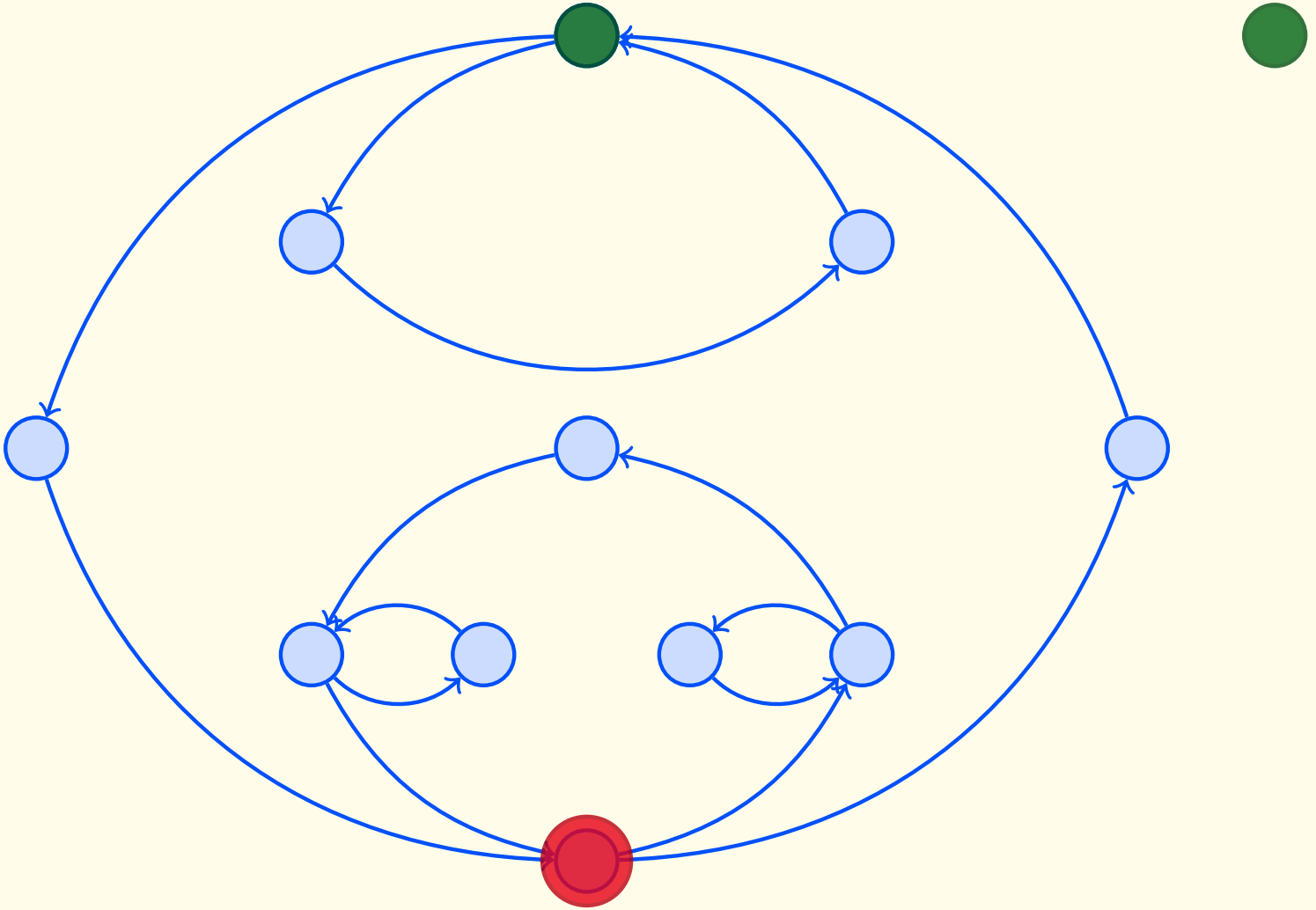

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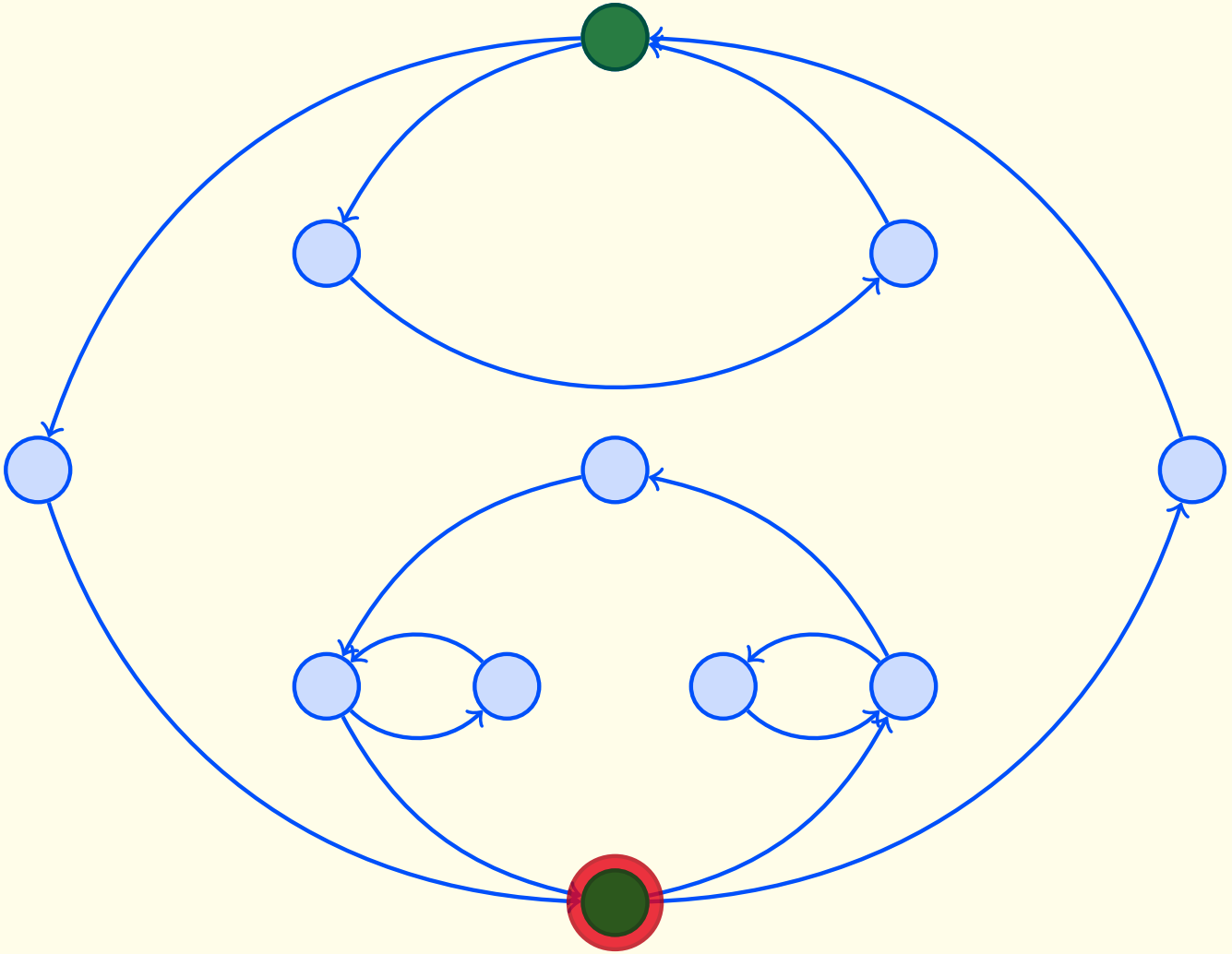

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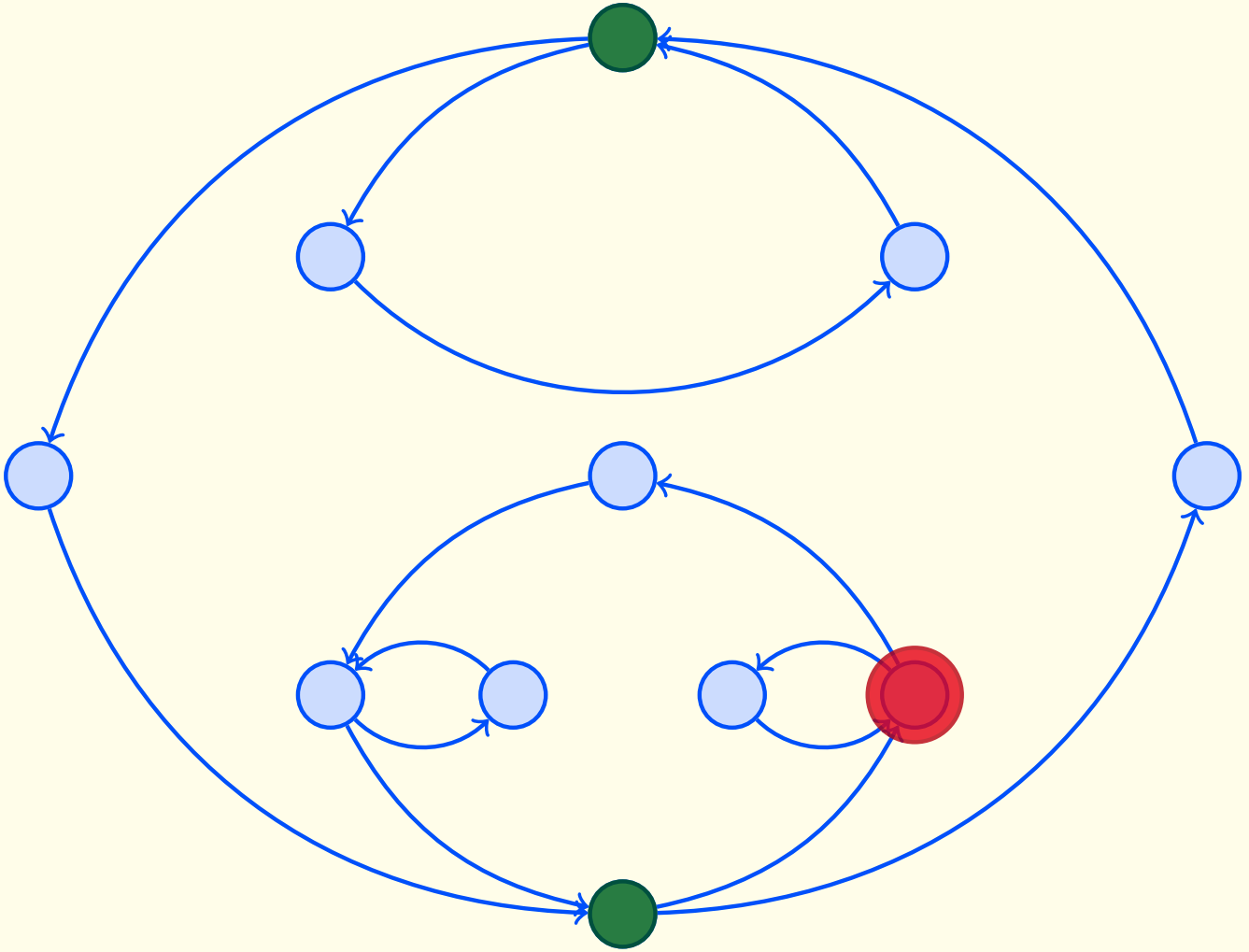

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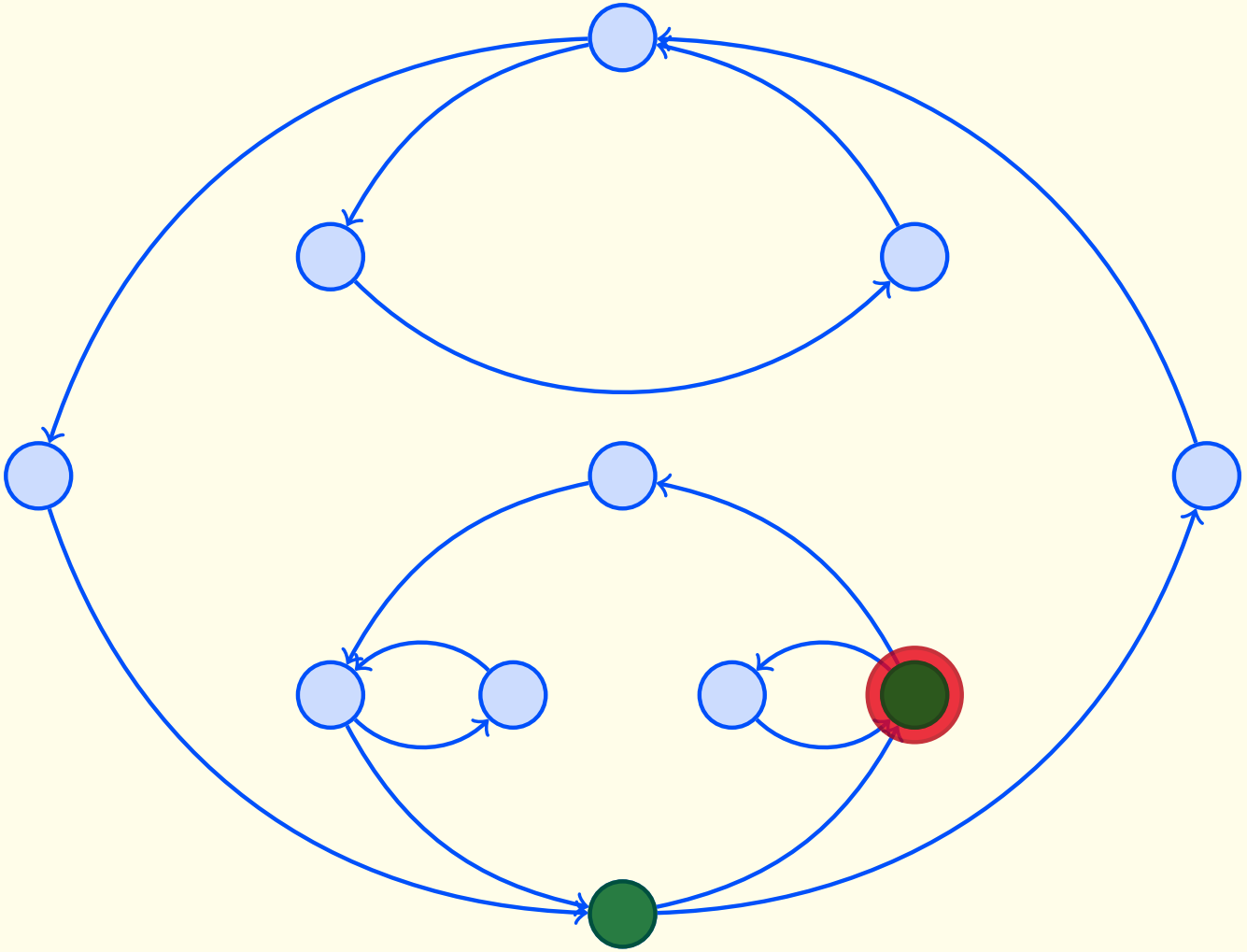

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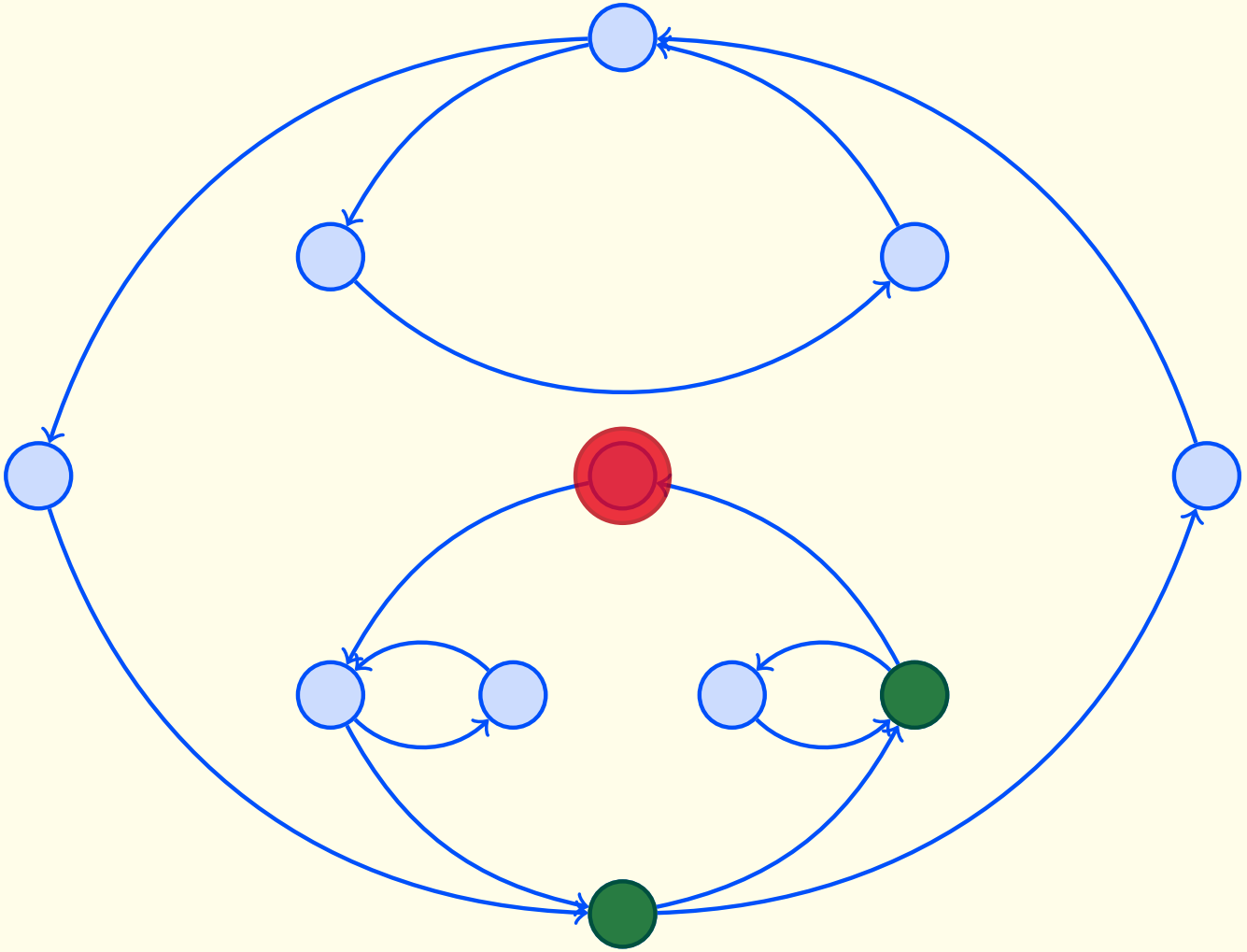

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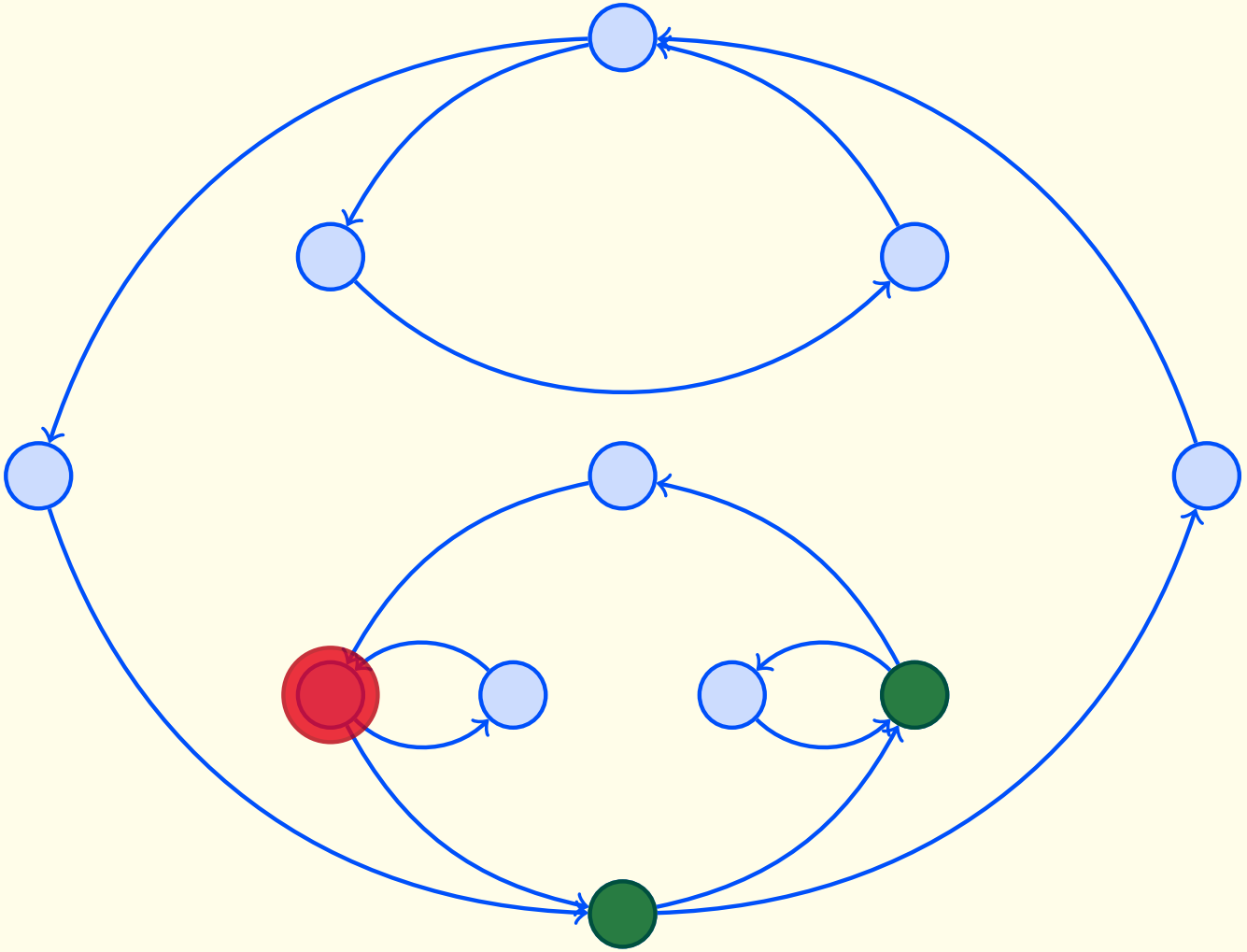

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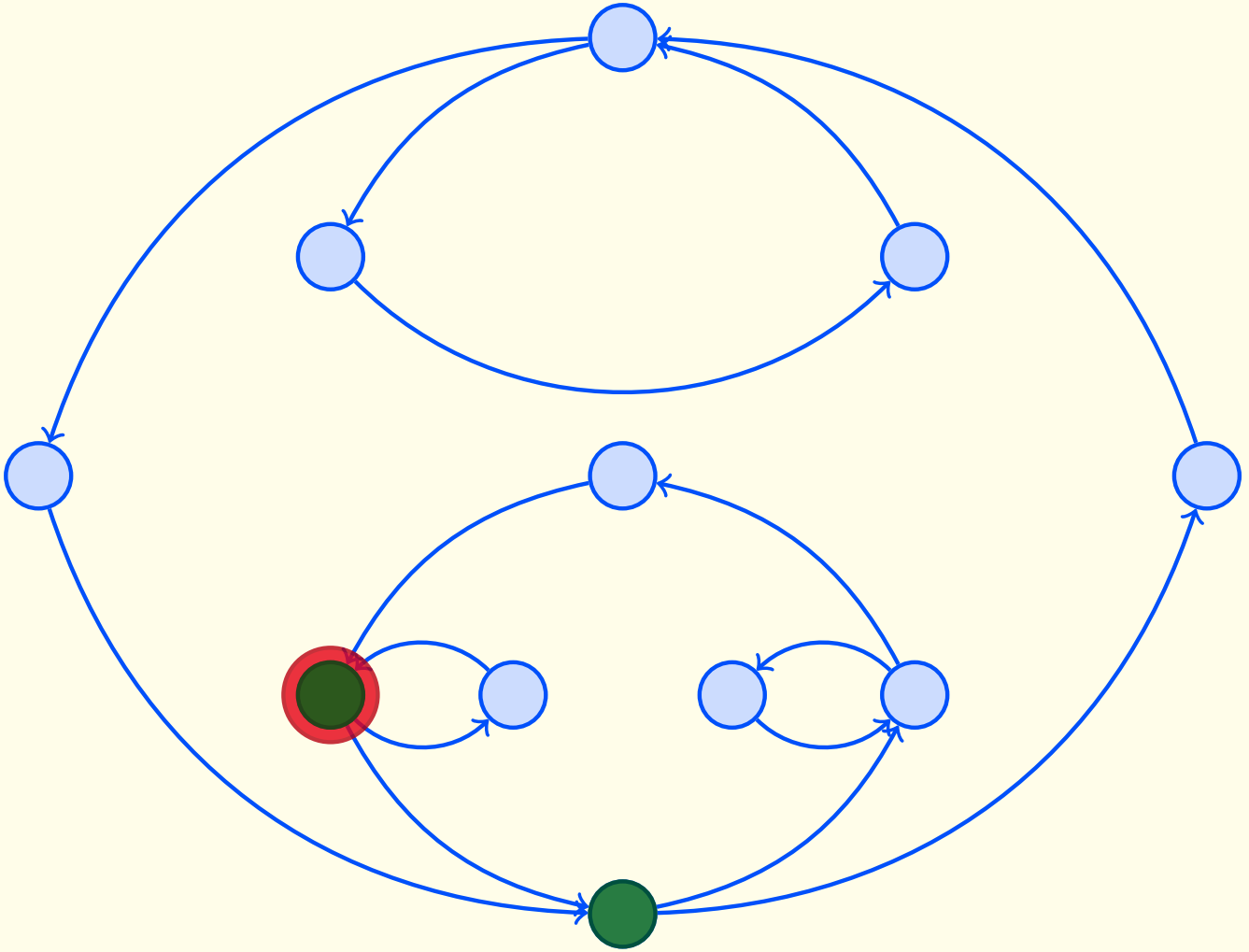

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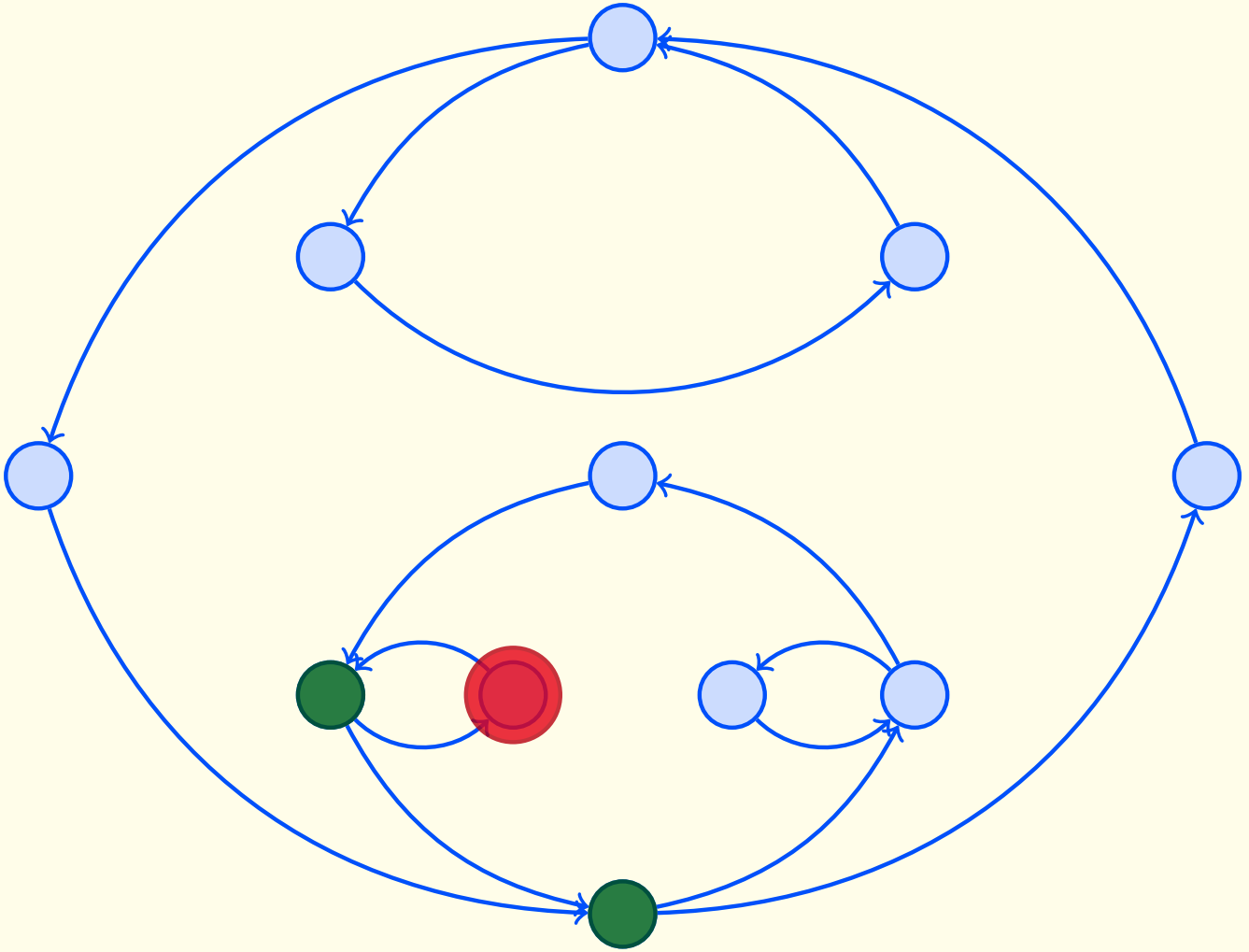


Undirected Graphs of Entanglement Two

- Belkhir, Santocanale: a tree-like decomposition for *undirected* graphs
- We generalise this result to *directed* graphs.

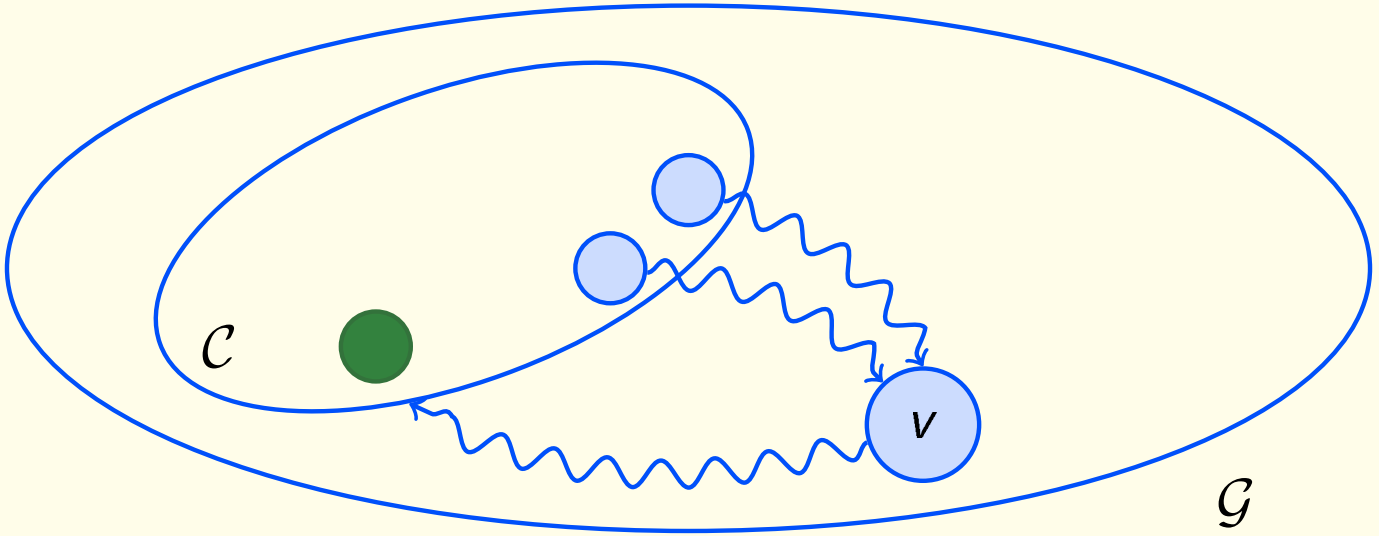
Simple and Complex Subgraphs

v a vertex
$\mathcal{C}$: a v-component of $\mathcal{G}$: an SCC of $\mathcal{G} \backslash v$

$\mathcal{C}$: is simple if

(1) $\text{entanglement}(\mathcal{C}) \leq 1$

(2) cops win the entanglement game with exit vertices



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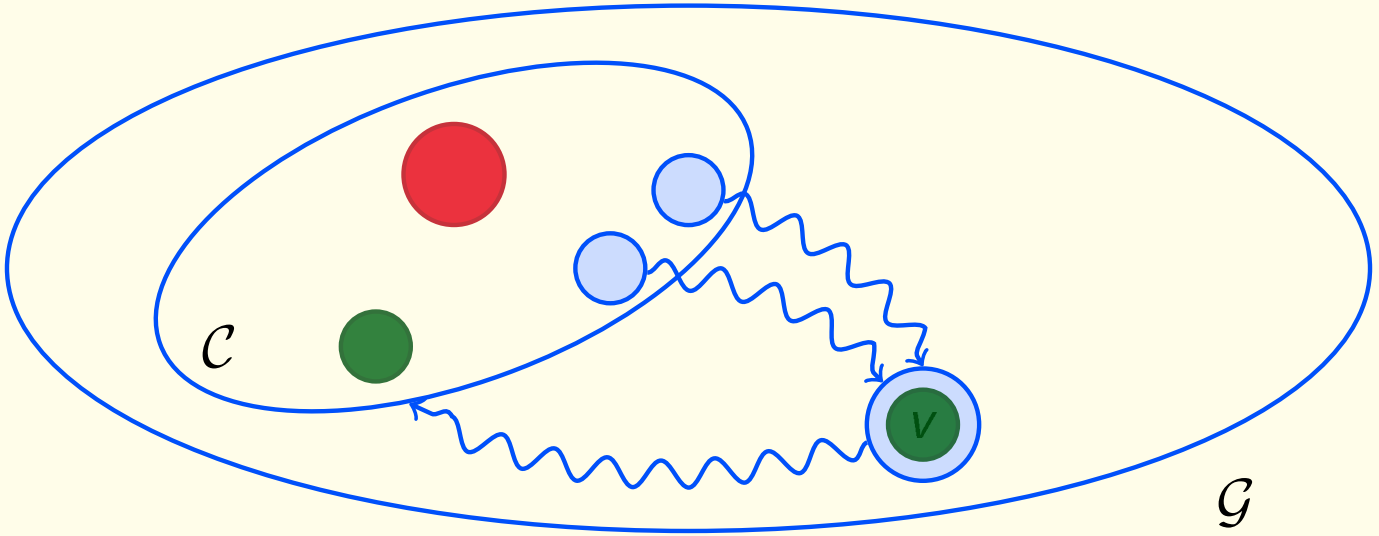


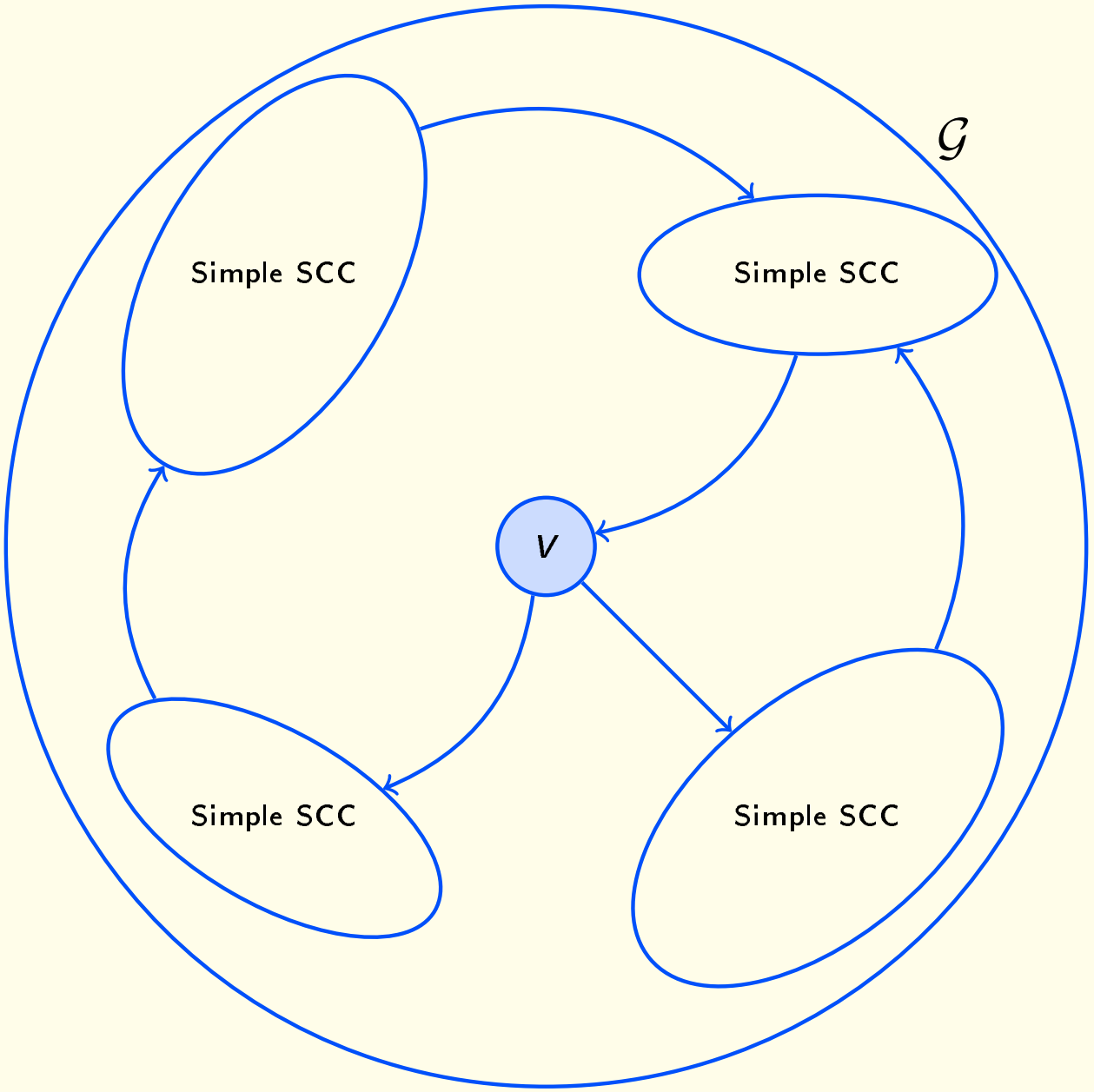
otherwise $\mathcal{C}$ is complex

Game Characterisation

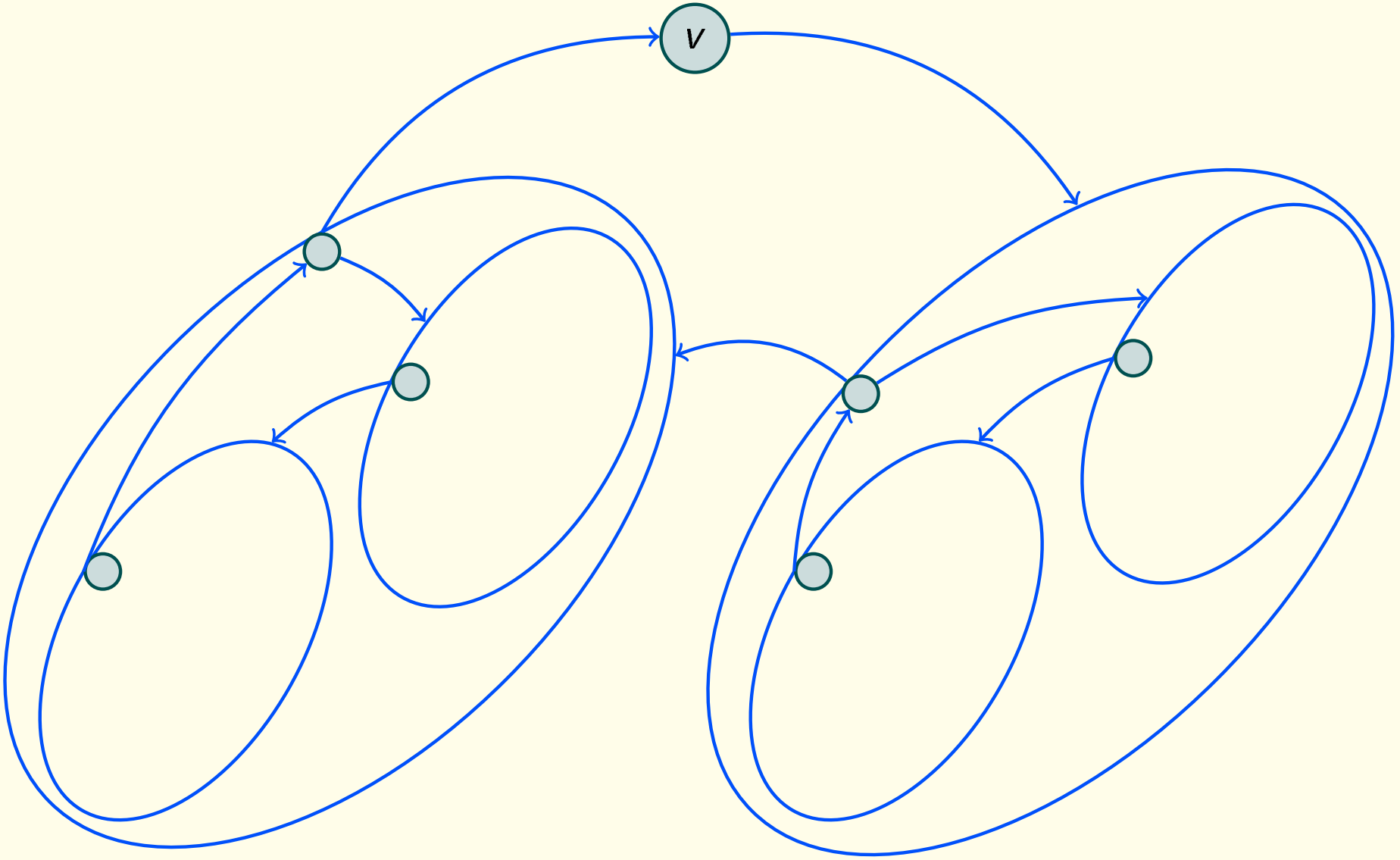
Theorem

On a strongly connected graph $\mathcal{G}$, two cops win the entanglement game iff there exists a vertex $v \in \mathcal{G}$ such that every SCC of $\mathcal{G} \backslash v$ is simple.

Game Characterisation



Decomposition Characterisation



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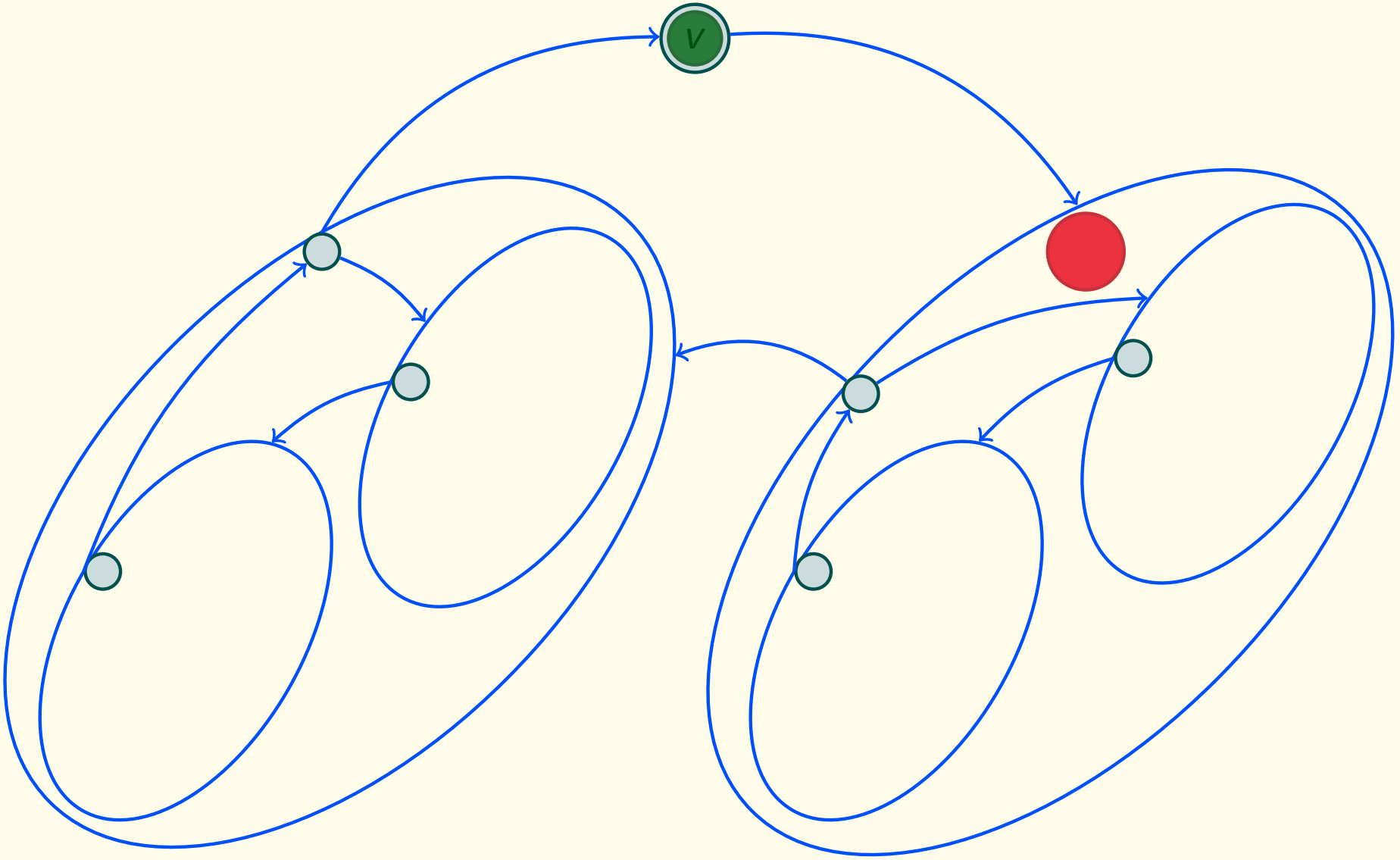

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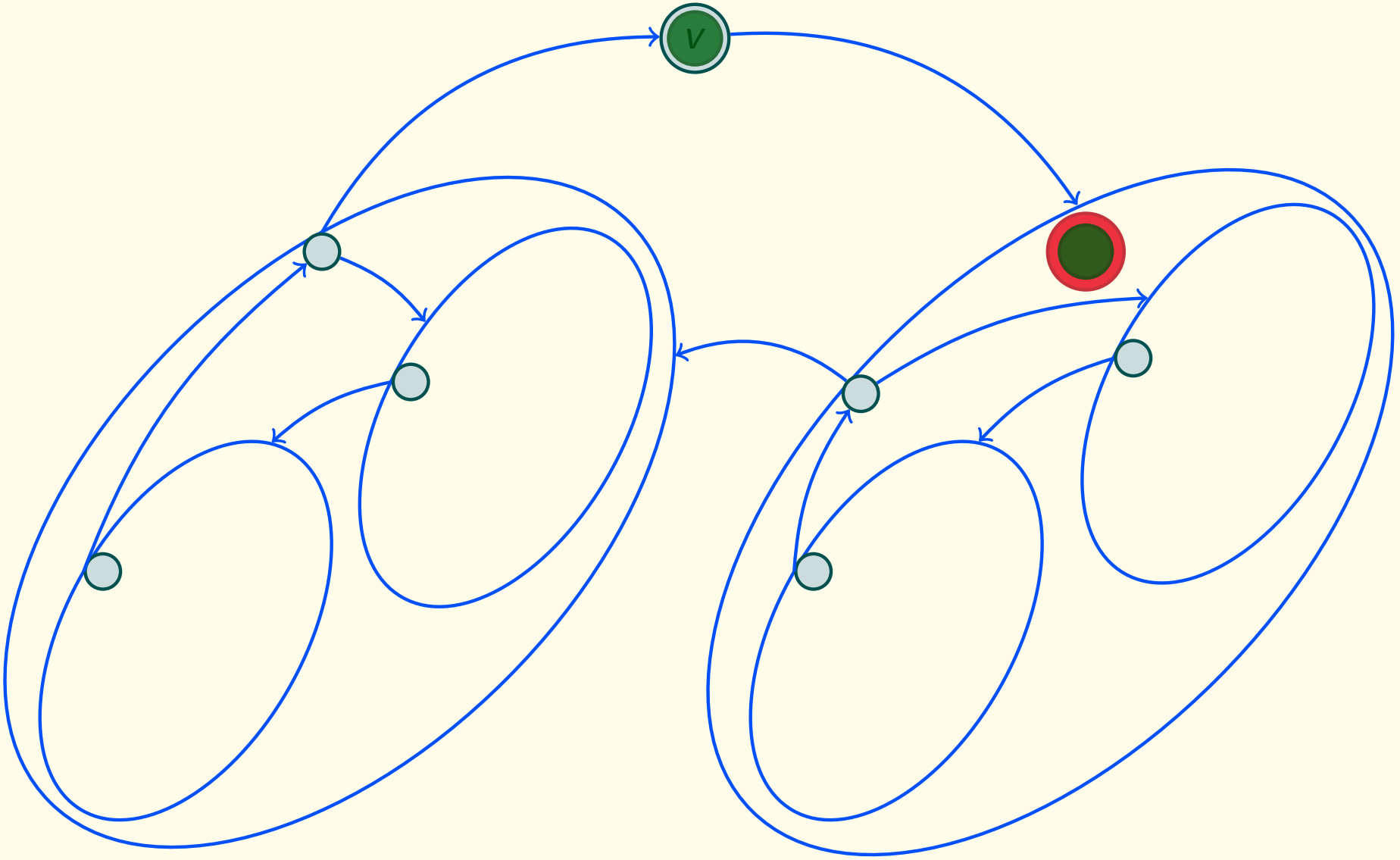

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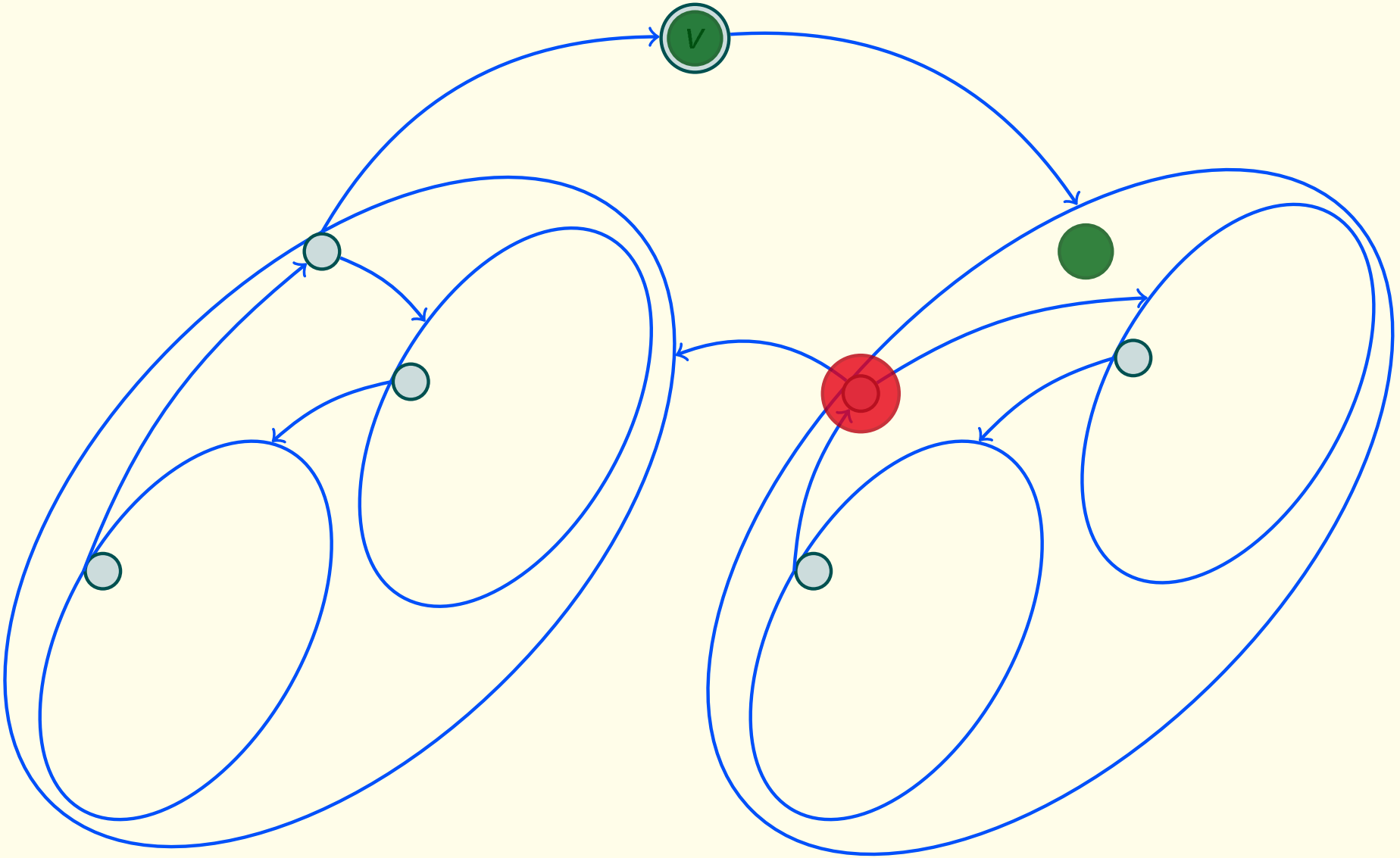

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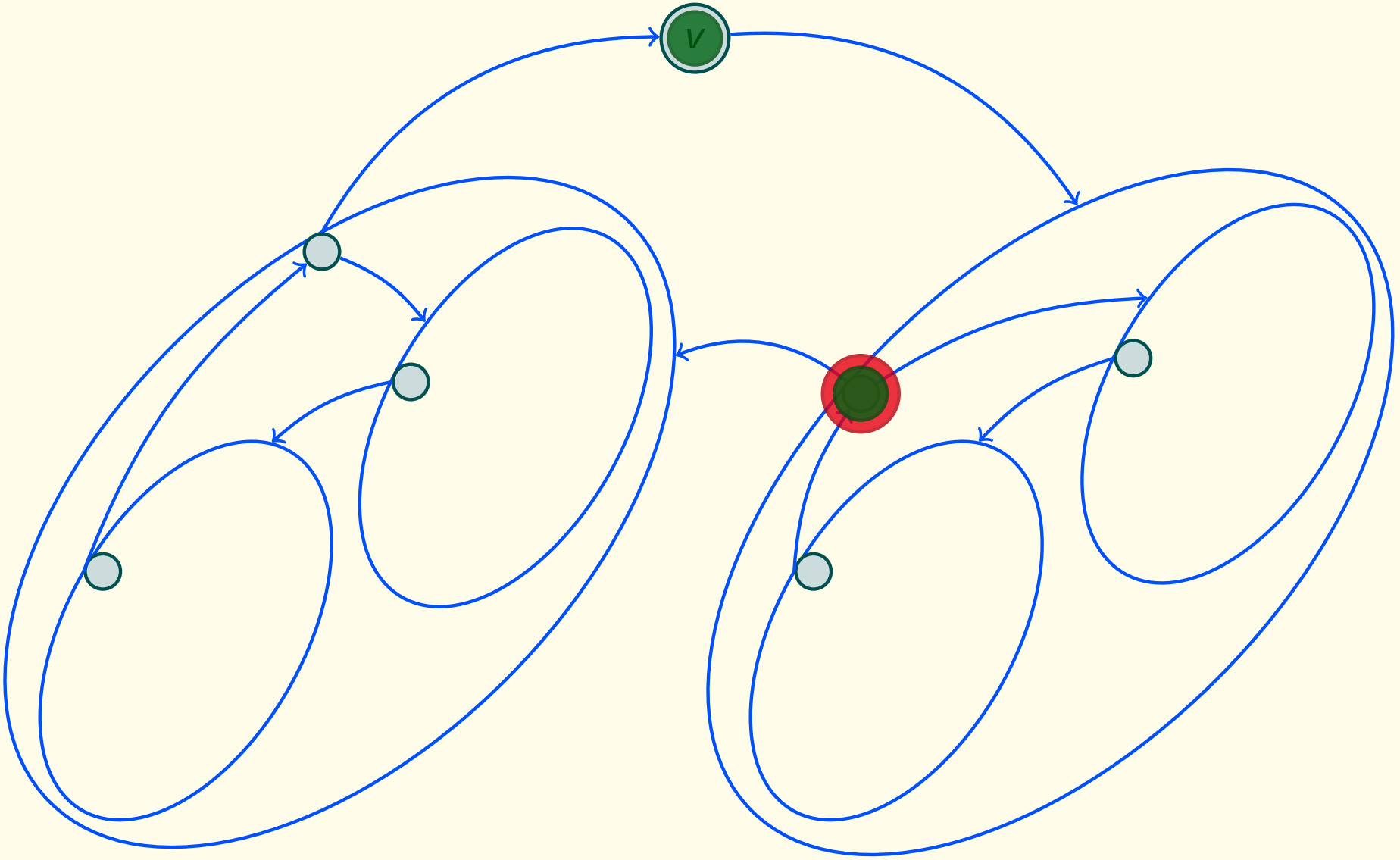

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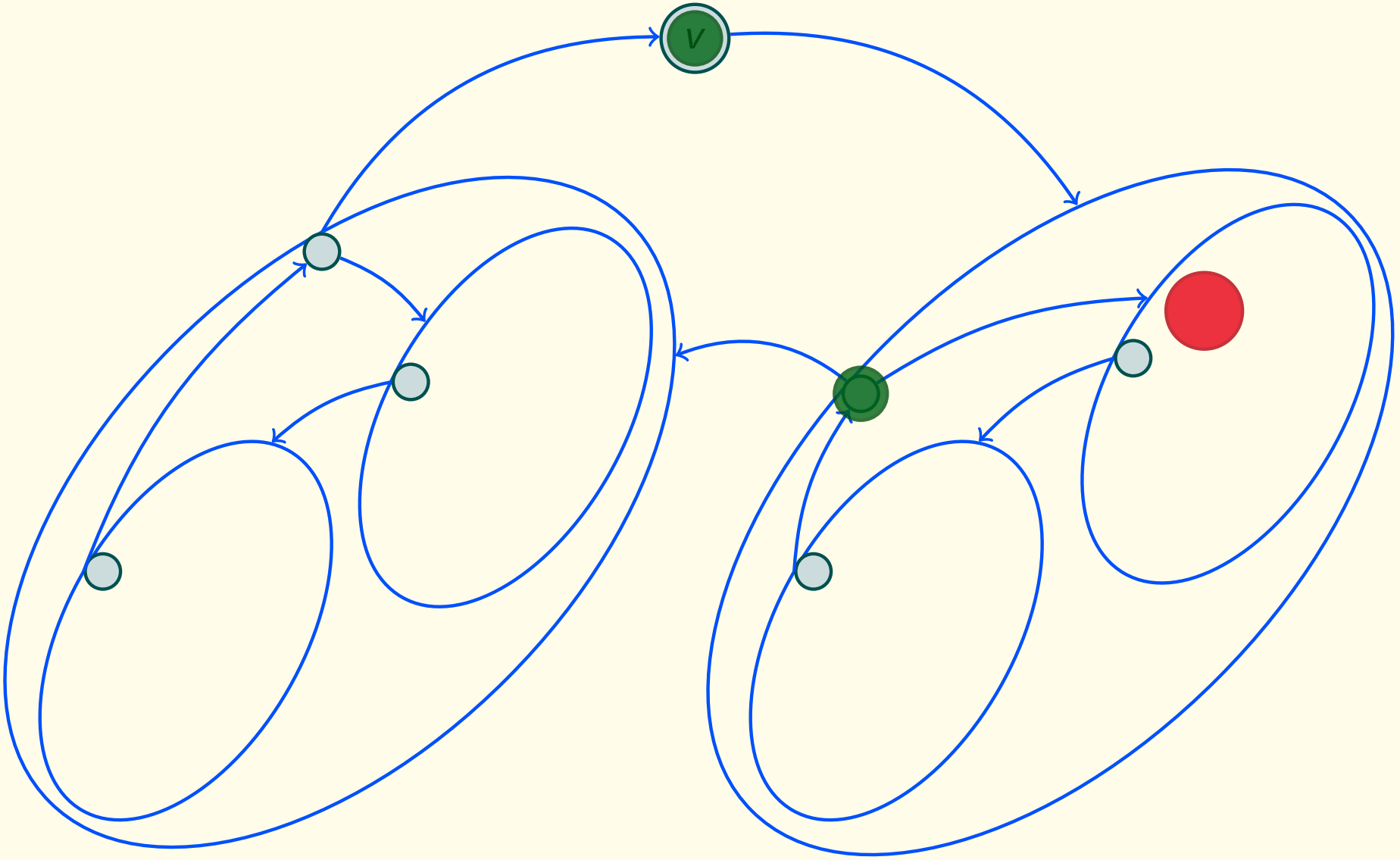

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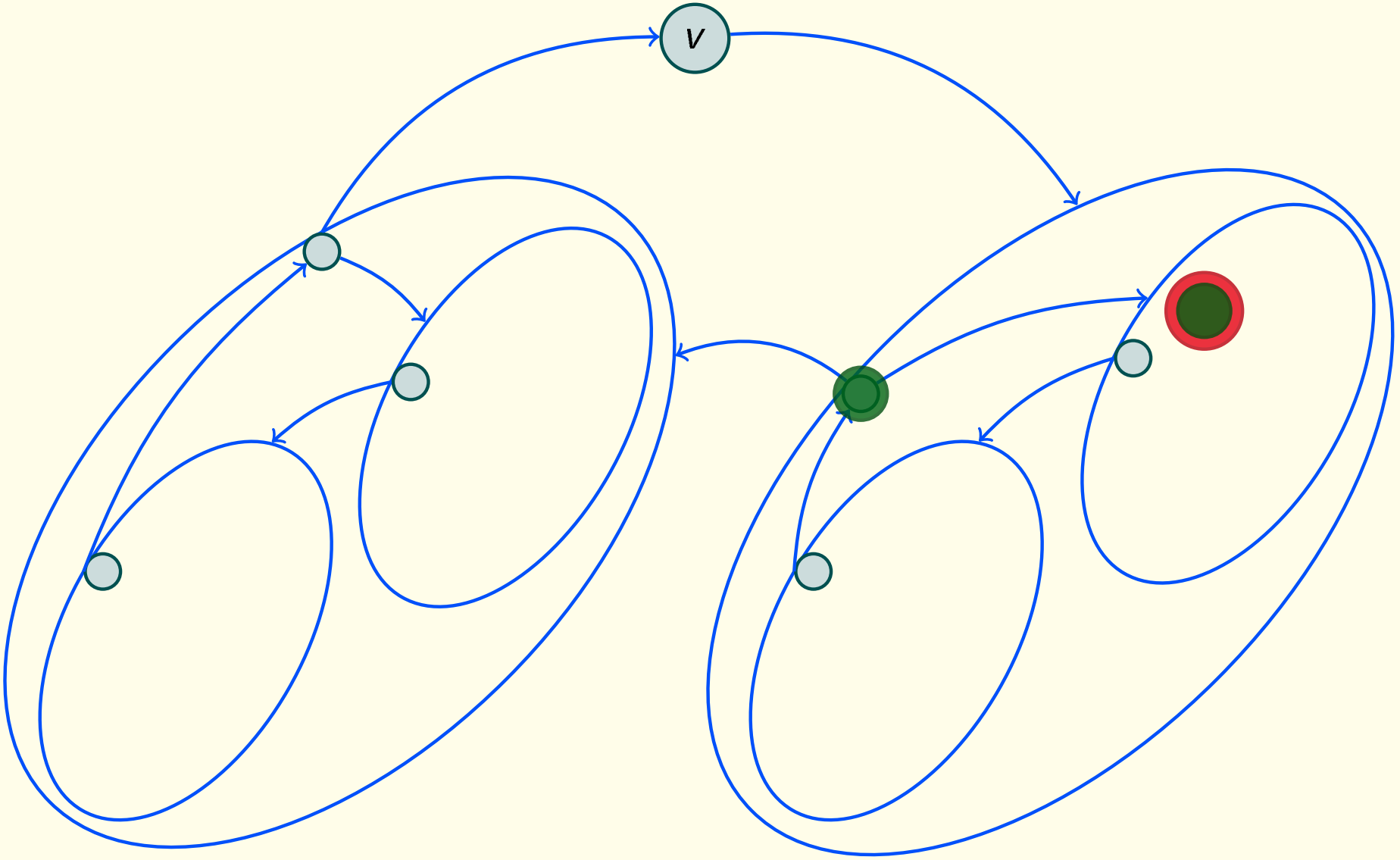

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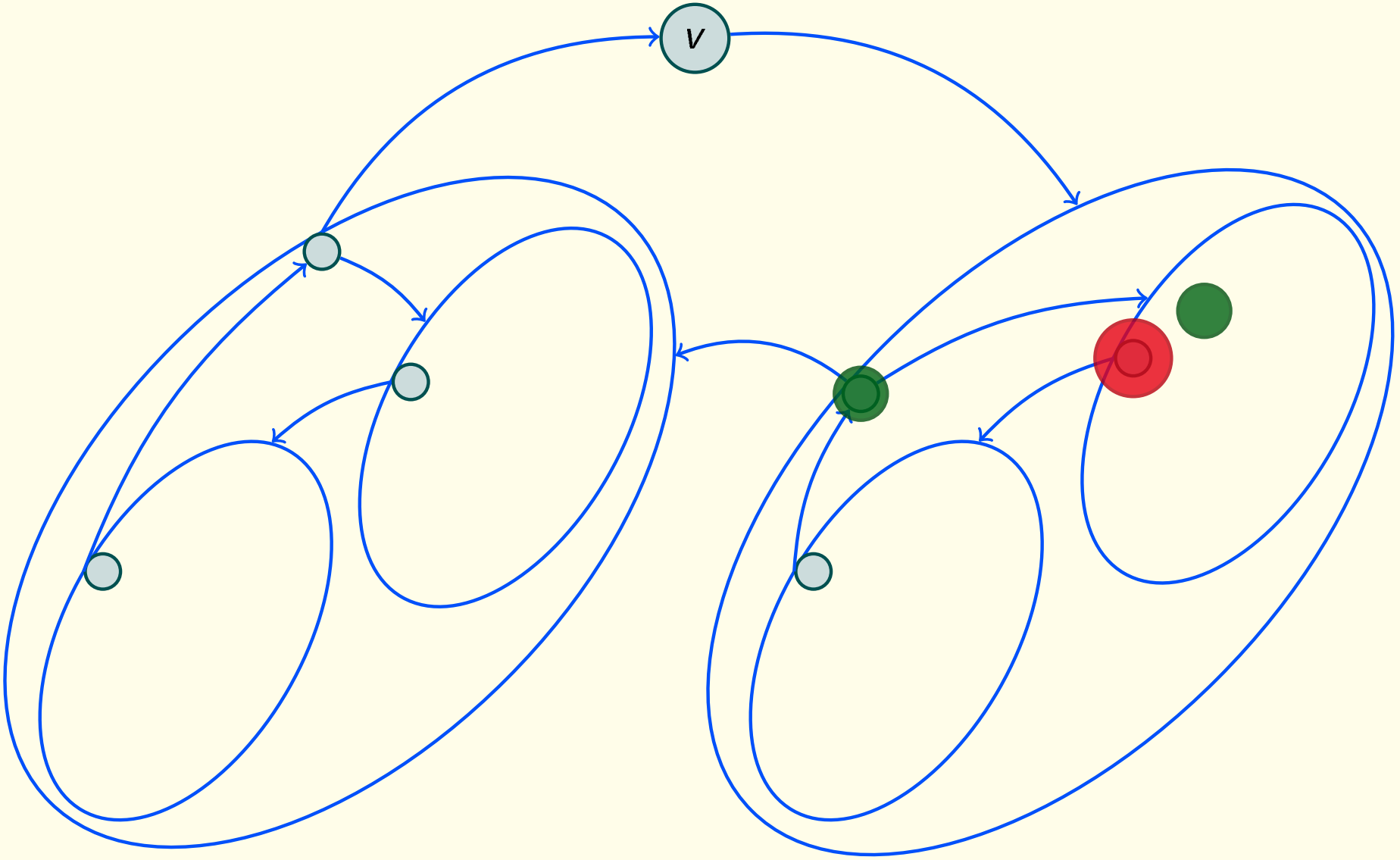

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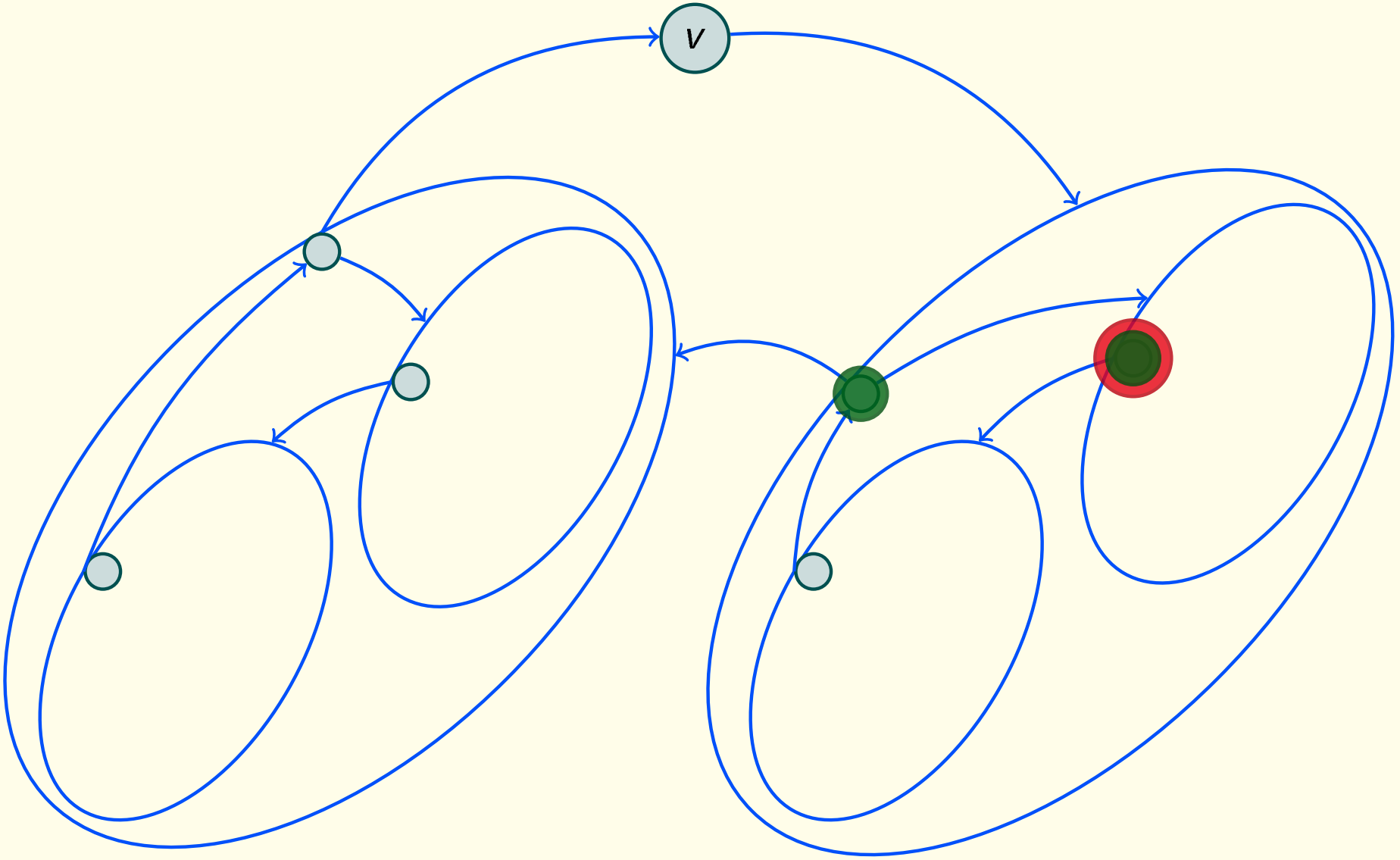

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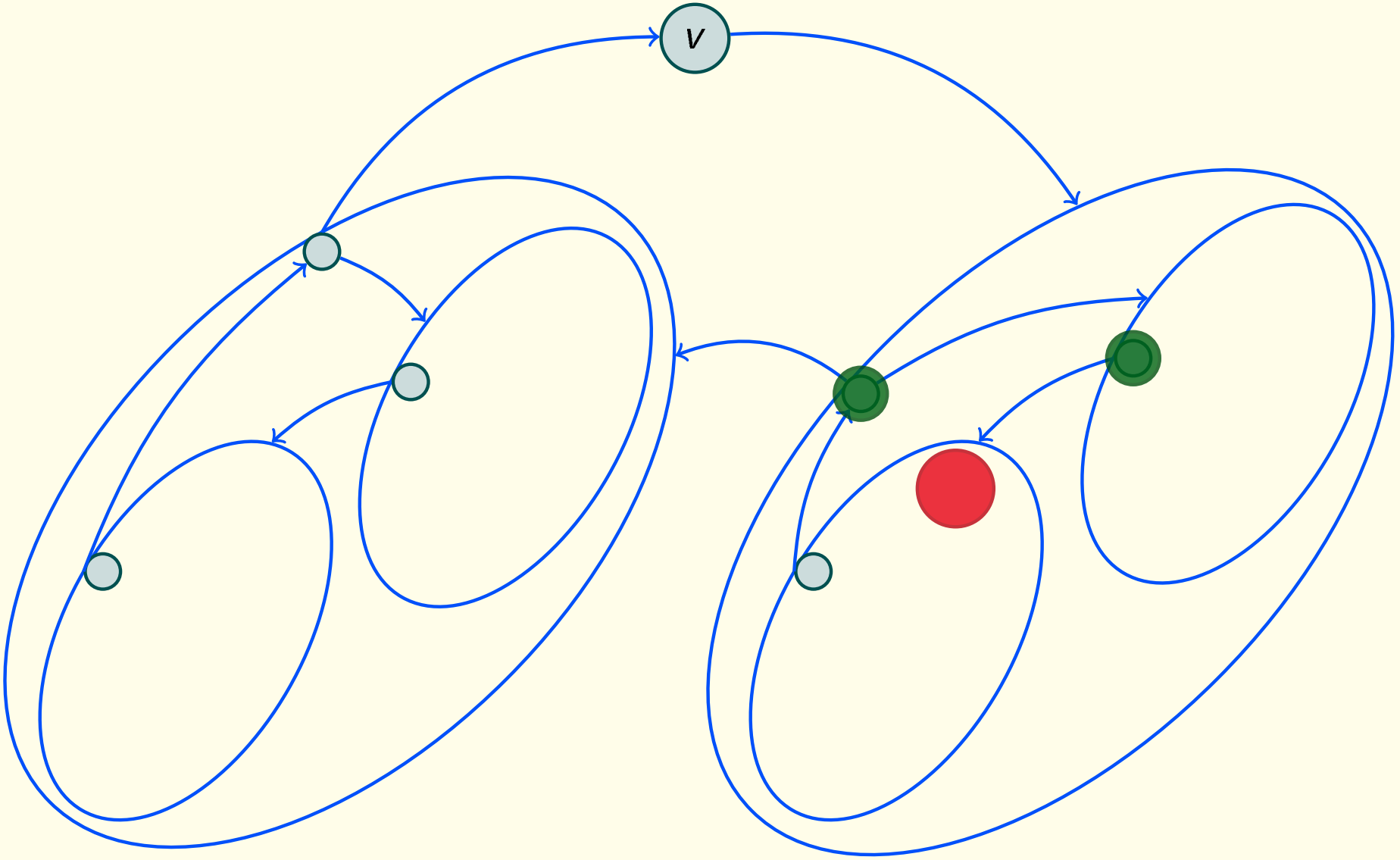
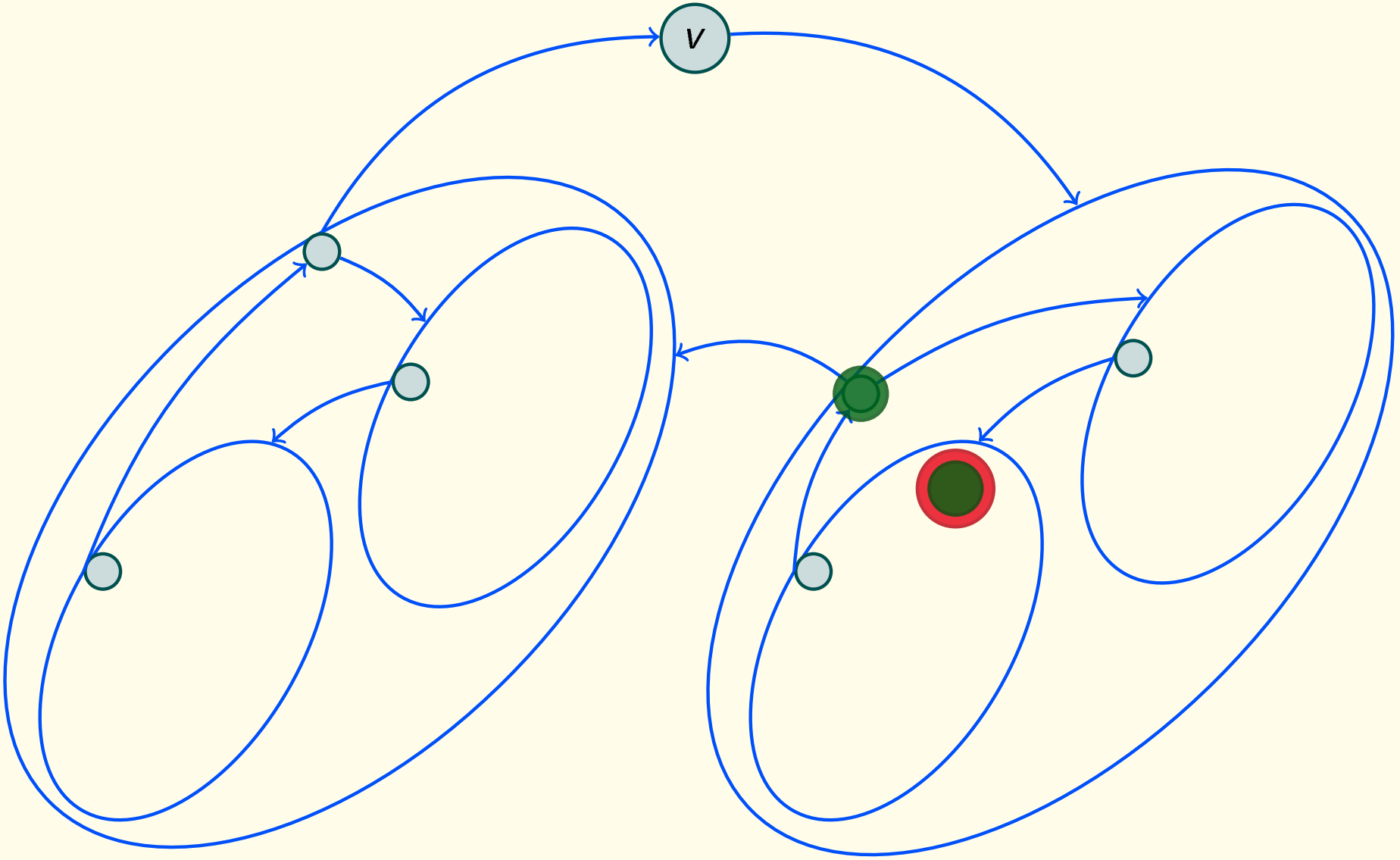
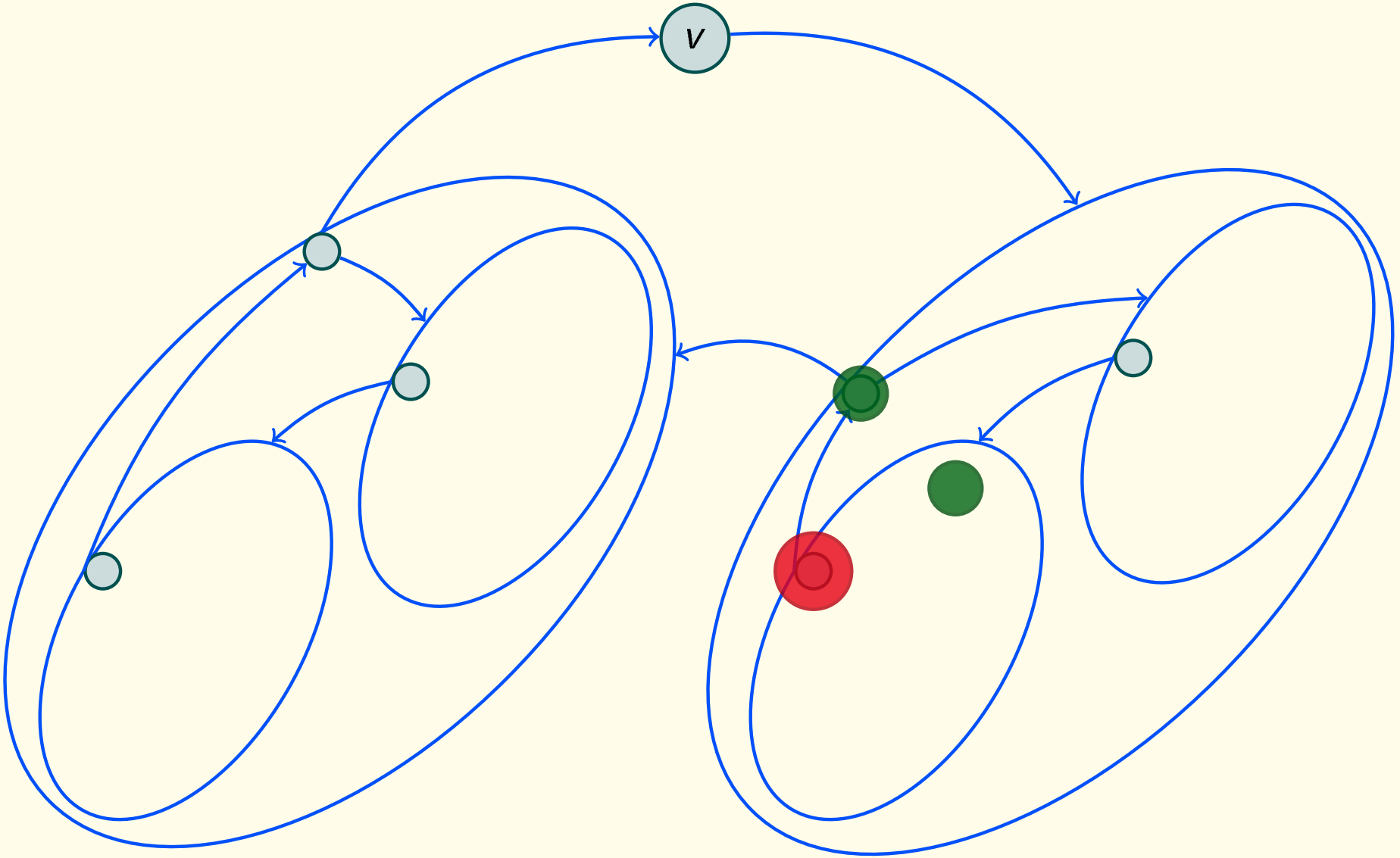


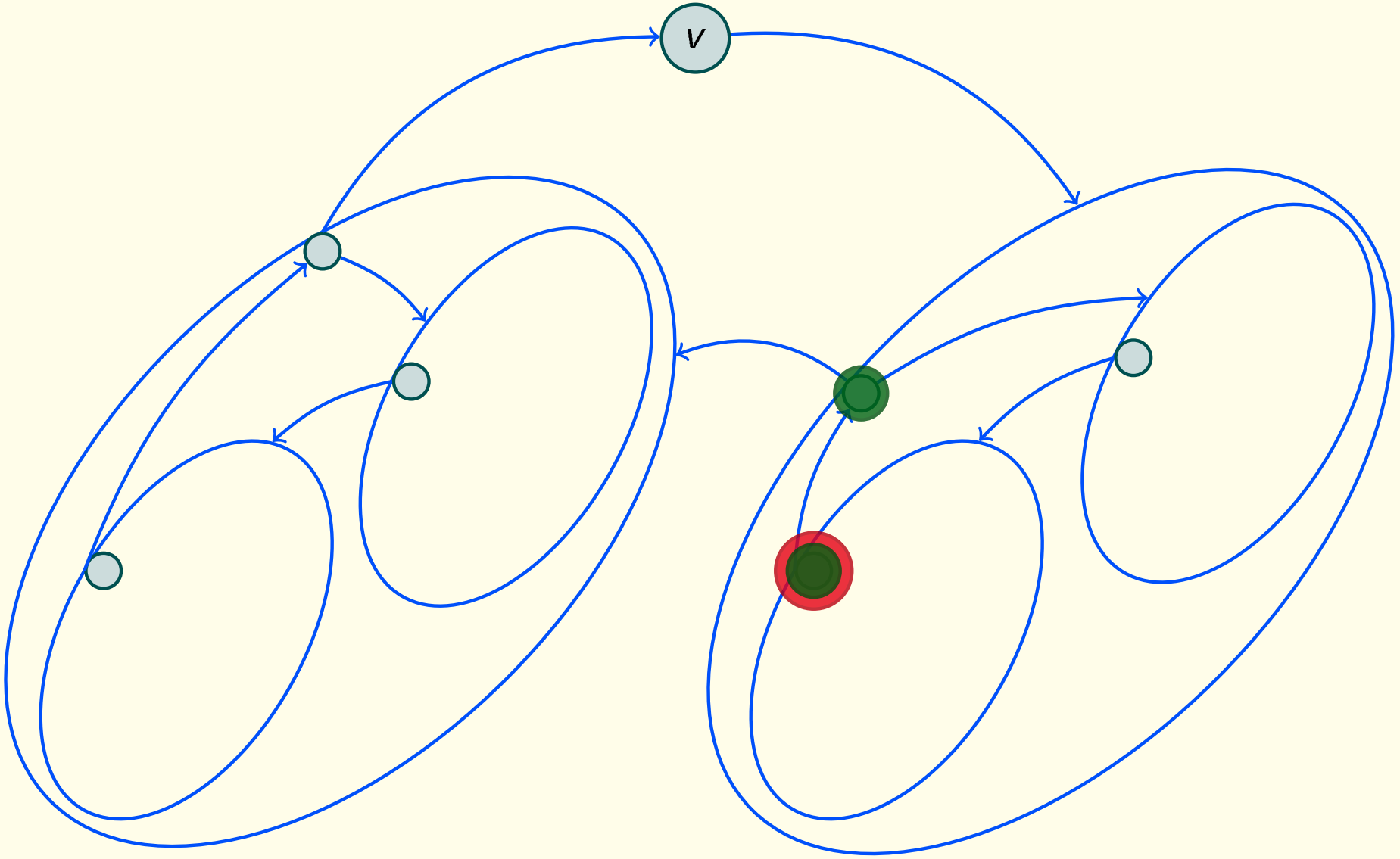
Decomposition Characterisation



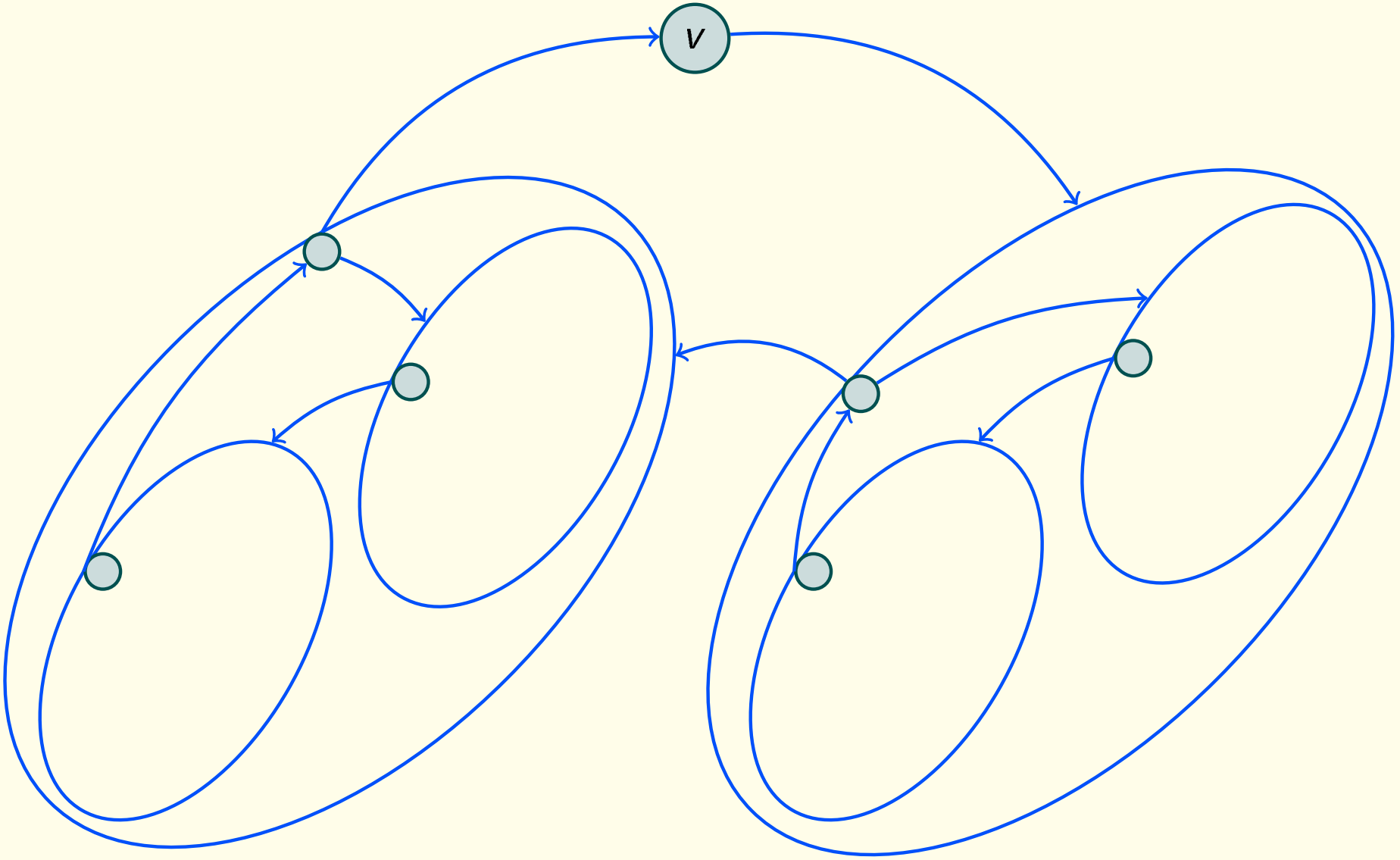
Decomposition Characterisation



Decomposition Characterisation



Inductive Construction



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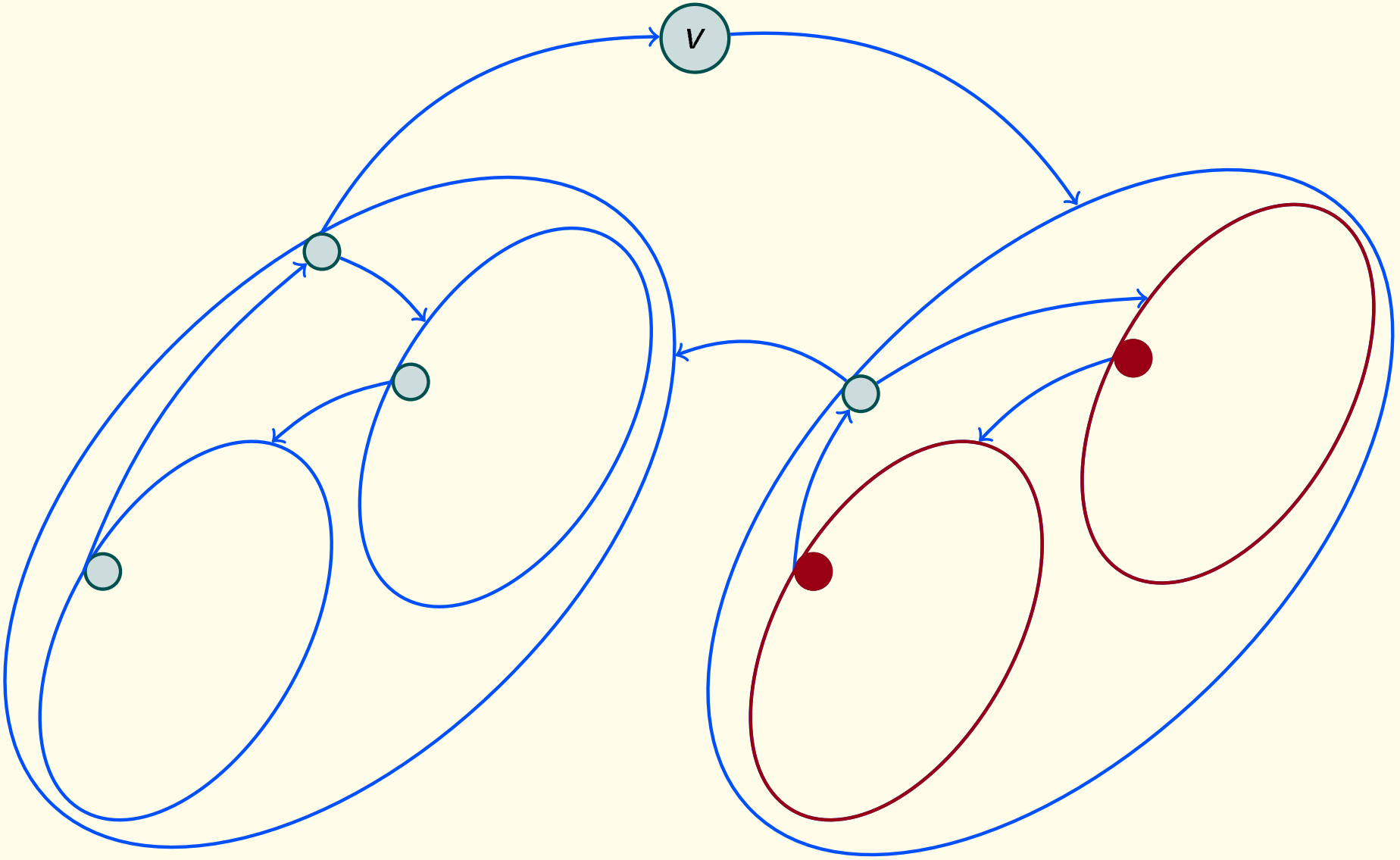

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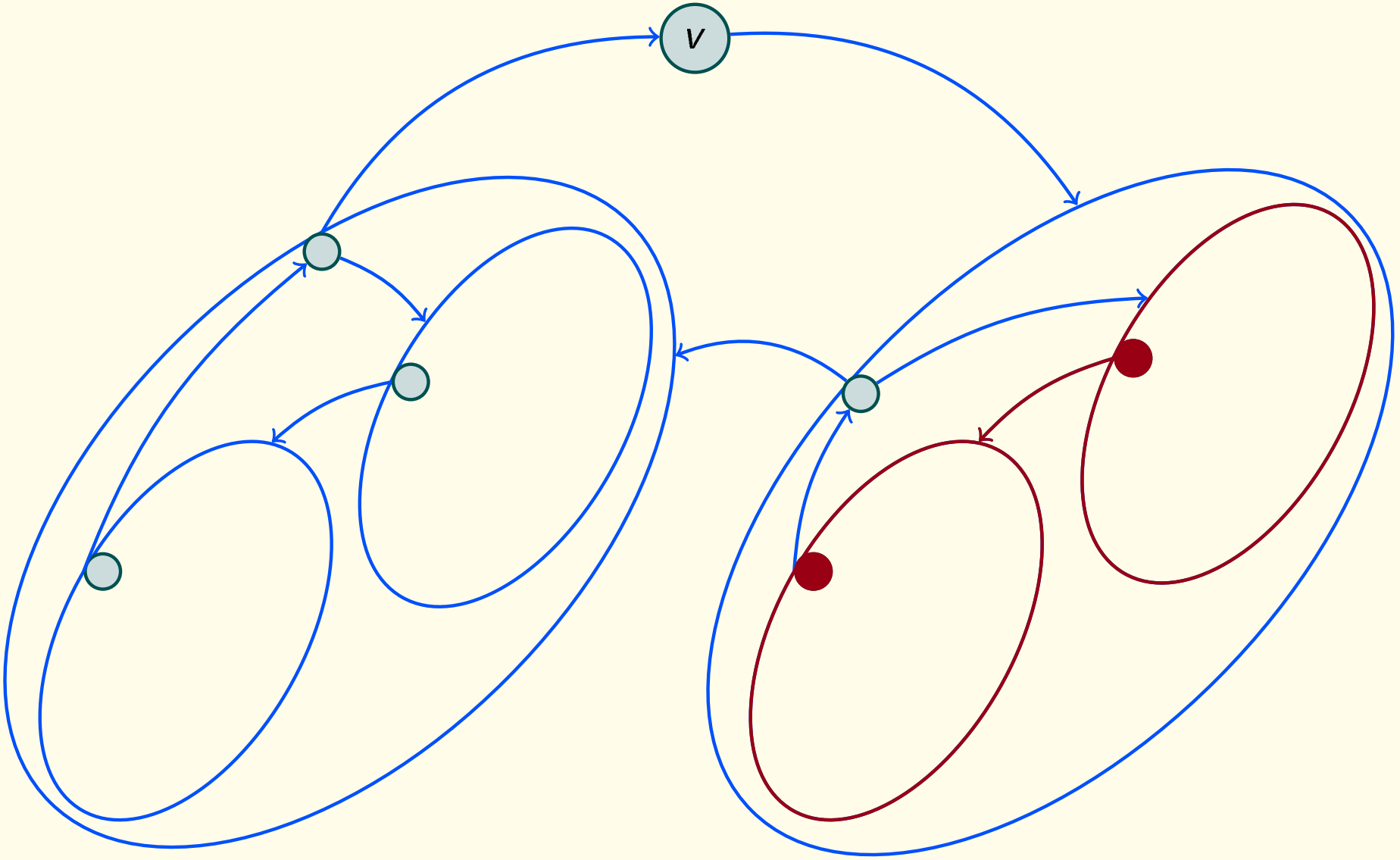

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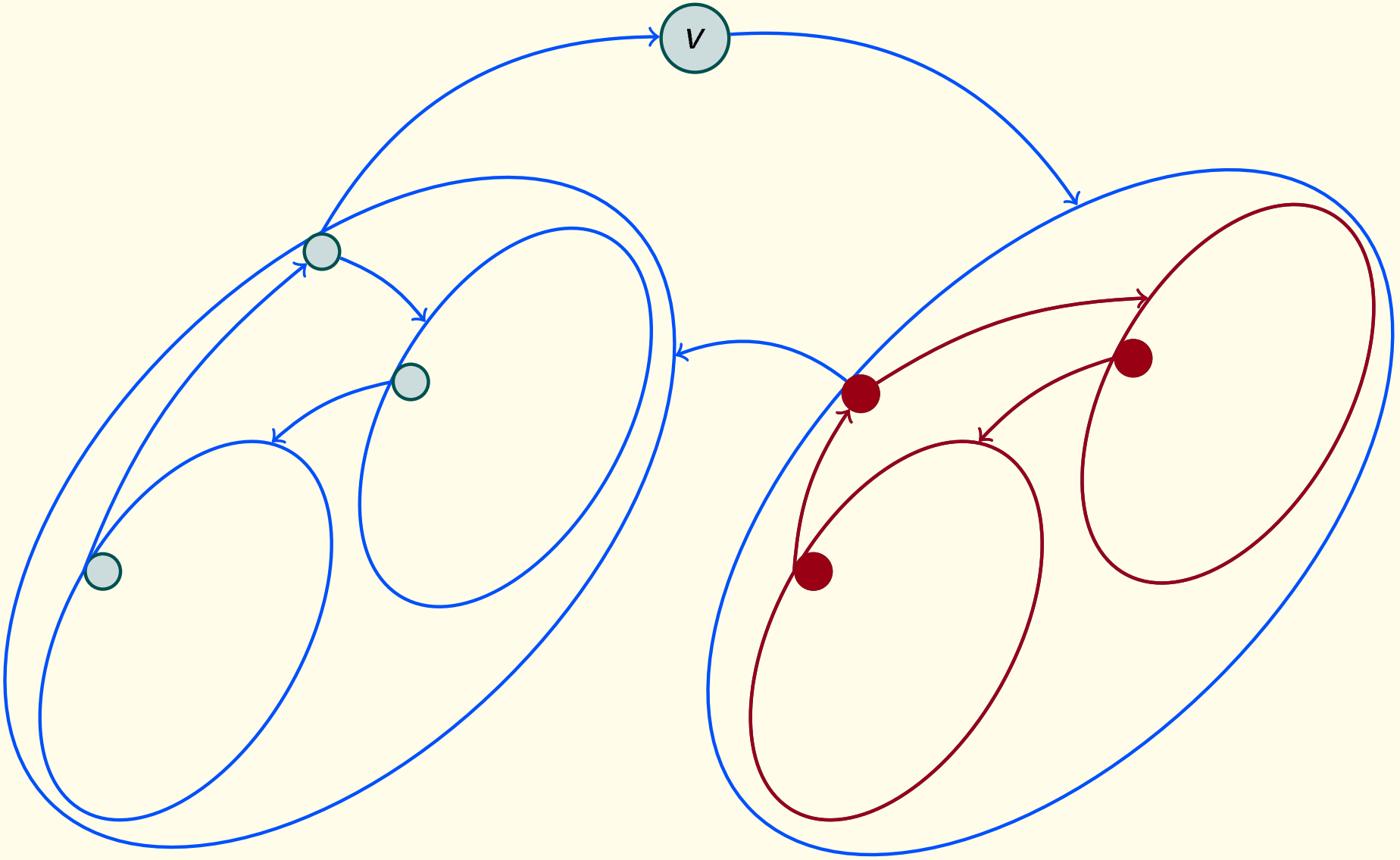

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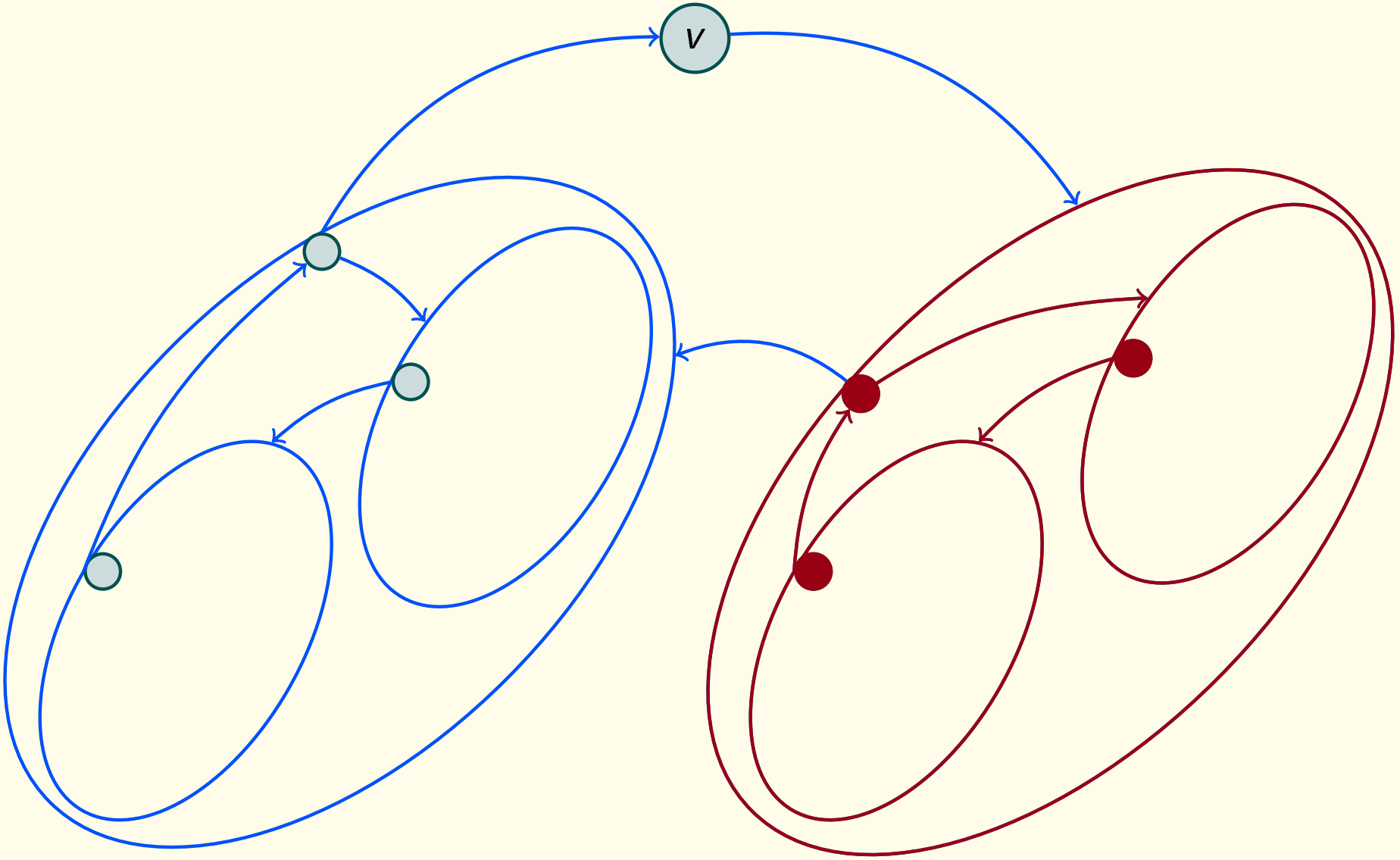

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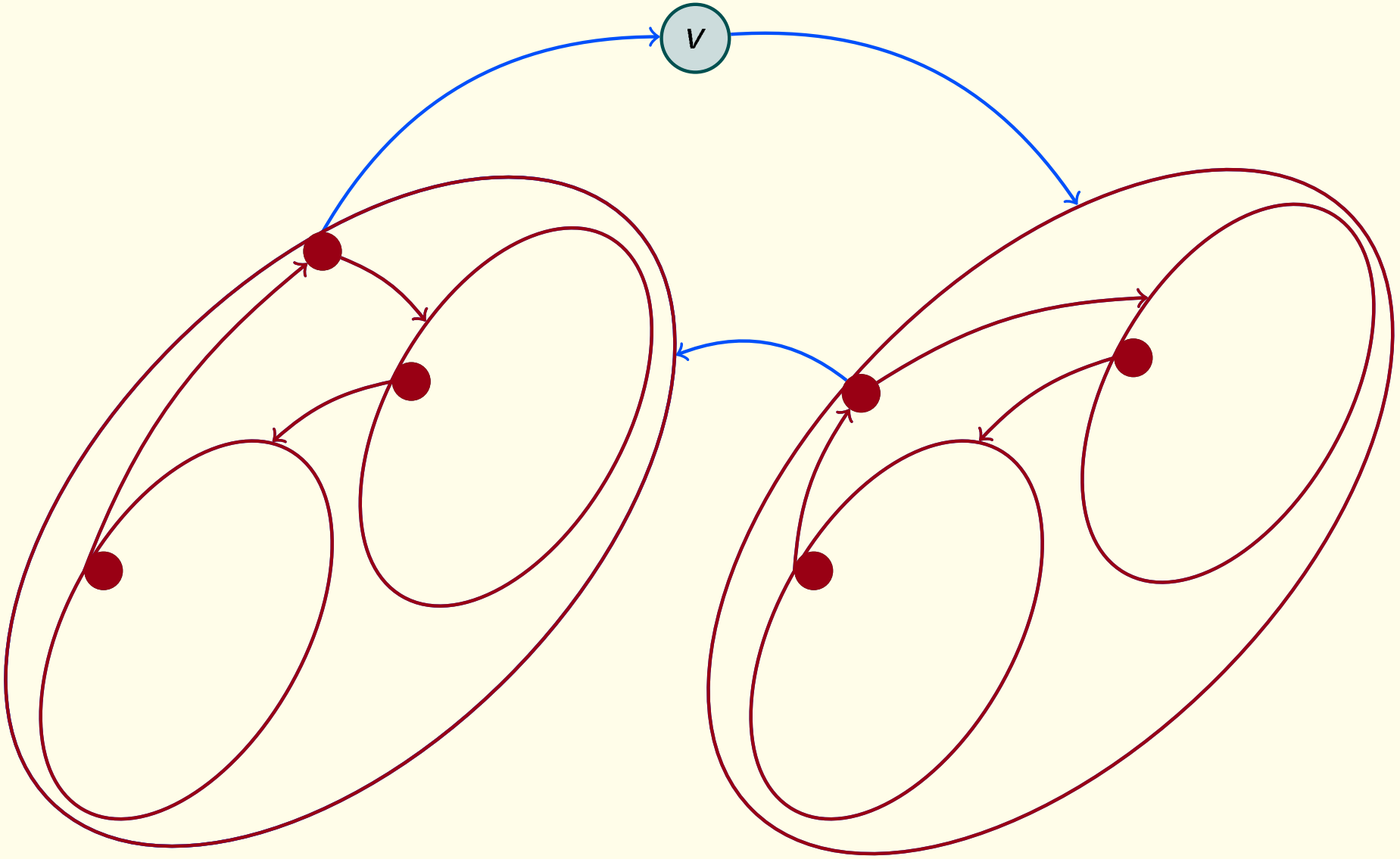

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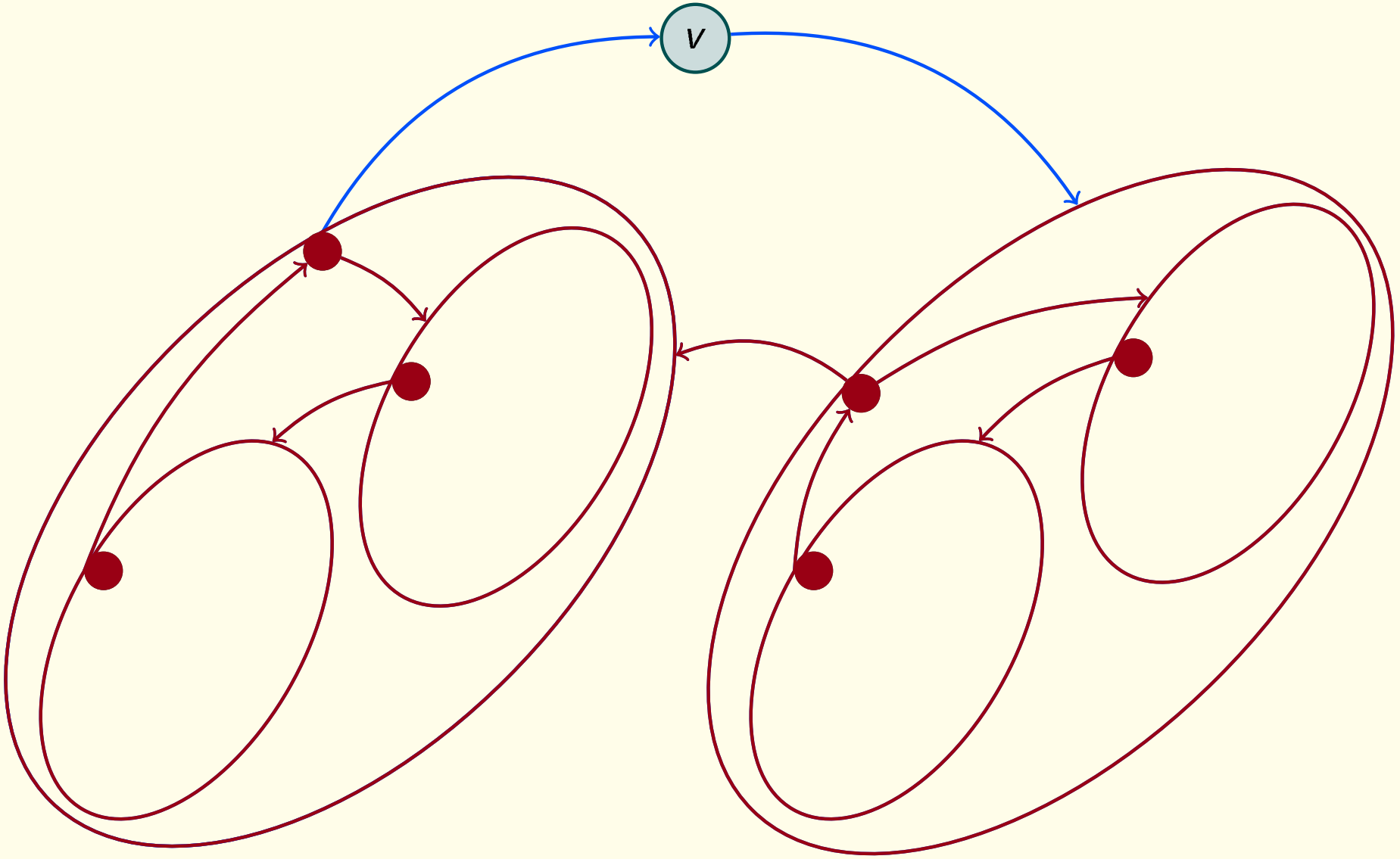
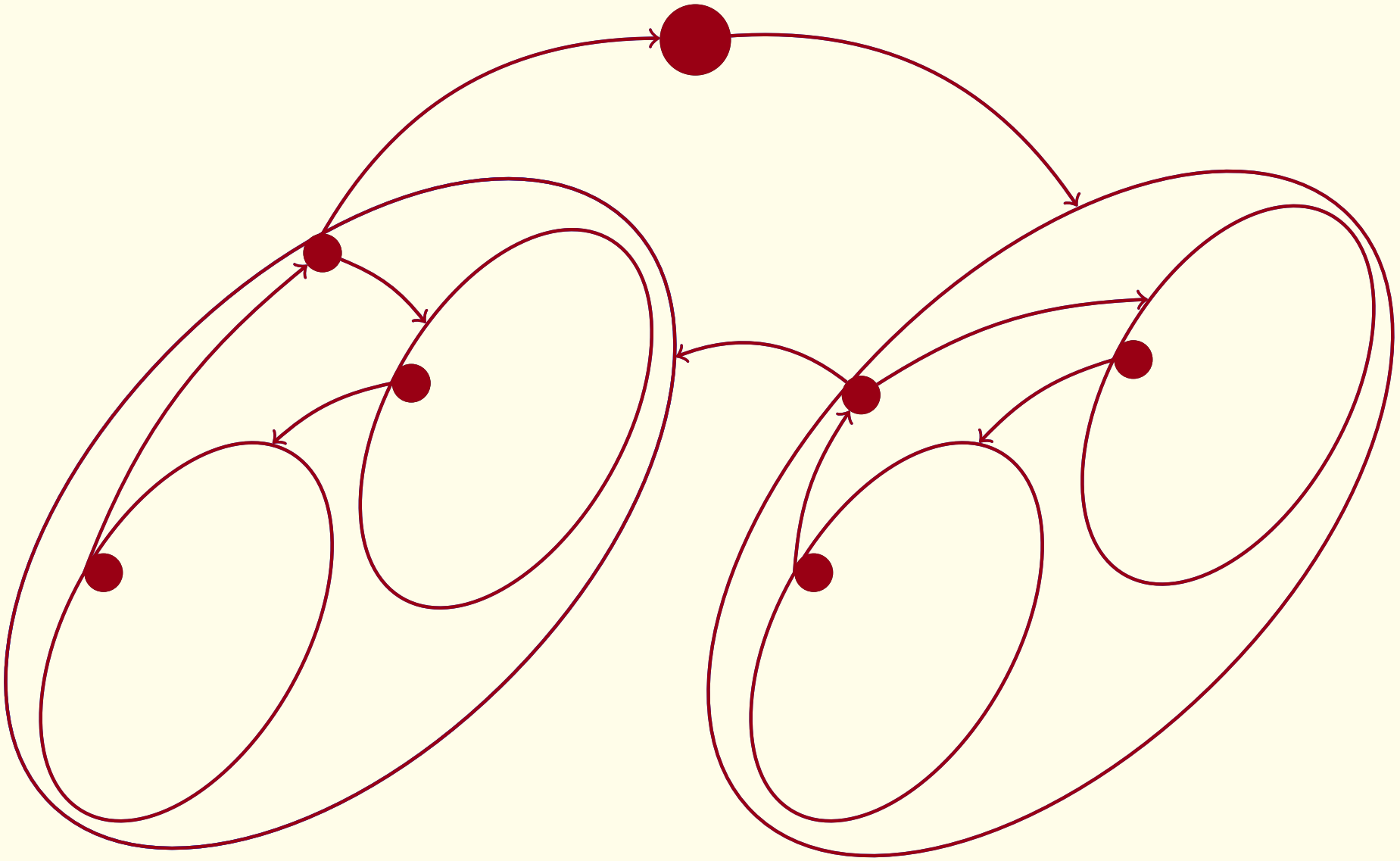


Inductive Construction



Failure of Generalisation and Future Work

- More than two cops: complex components do not work.
- $\Rightarrow$ Generalisation of complex components?

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Thank you!