RTN GAMES presents

Once Upon a Time in the West

(Determinacy, Definability, and Complexity of Path Games)

Actors: Ego

Alter

Story: Dietmar Berwanger, Erich Grädel, Stephan Kreutzer

Pictures: Eva Gajdos

Once upon a time in the west,. . .

. . . two players, Ego and Alter, set out for an infinite ride.

They were not exactly friends, and each of them would try to get the best

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Consequently, they would never agree on the route to take . . .

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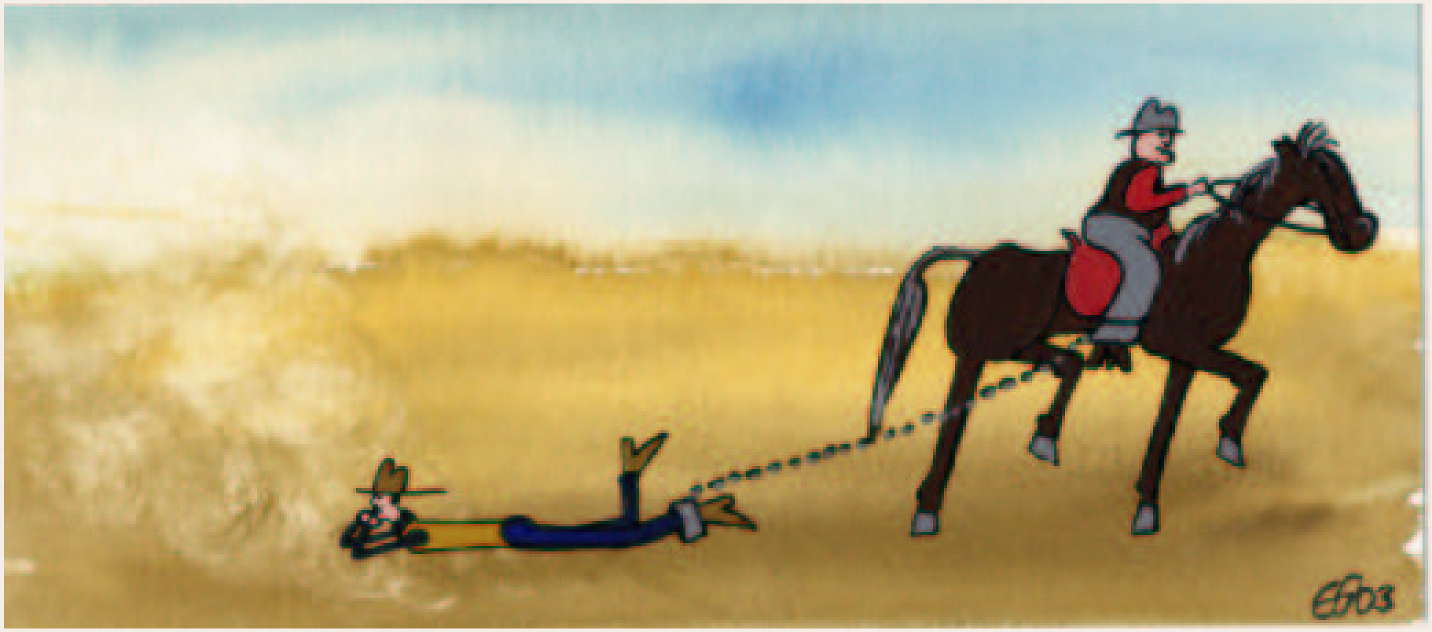
forced them
to stay together.



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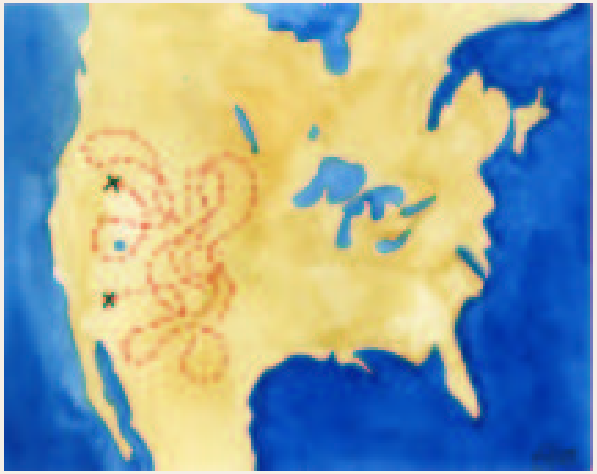

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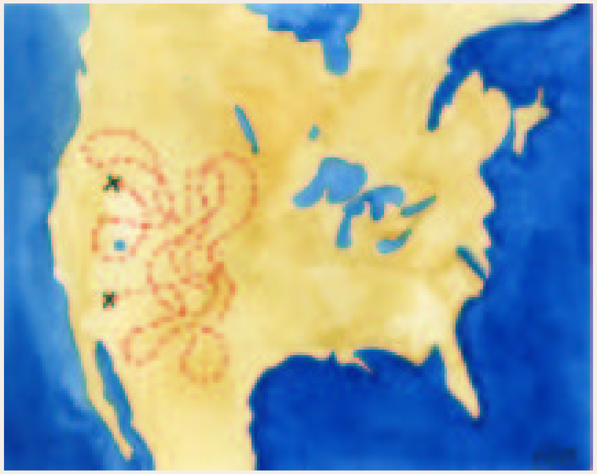
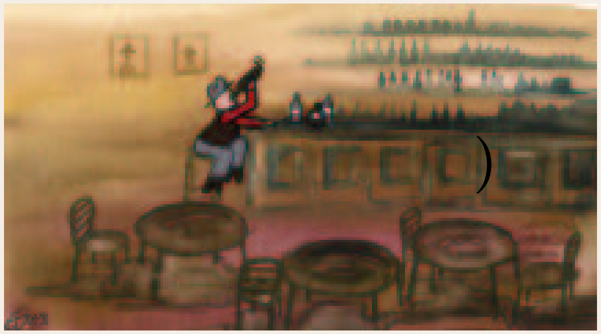


or take it easy

and make sure the day ends well



)

After ω days an infinite ride is completed and it is time for payoff!

Each player gets what he was able to collect during the ride. So his payoff depends on what they have seen, the infinite path they took!

(One of the reasons why they could never agree on the route.)

Path games with finite alternations

There were variants of this game, where after a certain number of days, one of the players would be eliminated (these things happended in the west),

and the other would complete the game alone (a very lonesome ride indeed)

Path games

Arena: (G, W) consisting of a game graph $G = (V, E, v_0)$ and a winning condition W: a set of infinite paths from v_0 through G

Players: Ego (E) and Alter (A)

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Winning the game: After ω moves, an infinite path π is completed. Ego has won if $\pi \in W$, otherwise Alter has won.

Game prefixes

A game prefix $\gamma \in \{E, A\}^\omega$ indicates who begins and how many alternations are played. Obviously, $EE \equiv E$ and $AA \equiv A$. Hence, for any arena (G, W) we have the following games:

- $(EA)^\omega(G, W)$ and $(AE)^\omega(G, W)$:
  path games with infinite alternations
- $(EA)^k E^\omega(G, W)$ and $A(EA)^k E^\omega(G, W)$:
  games ending with infinite path extension by Ego
- $(AE)^k A^\omega(G, W)$ and $E(AE)^k A^\omega(G, W)$:
  games where Alter chooses the final infinite lonesome ride

Comparing path games

$\mathcal{G} \succeq \mathcal{H}$ means that $\mathcal{G}$ is better for Ego than $\mathcal{H}$:

$$\text{Ego wins } \mathcal{H} \quad \Longrightarrow \quad \text{Ego wins } \mathcal{G}$$

$$\text{Alter wins } \mathcal{G} \quad \Longrightarrow \quad \text{Alter wins } \mathcal{H}$$

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Obviously, $EAE^{\omega}(G, W) \succeq (EA)^{k}E^{\omega}(G, W)$.

But is $EAE^{\omega}(G, W)$ strictly better for Ego?

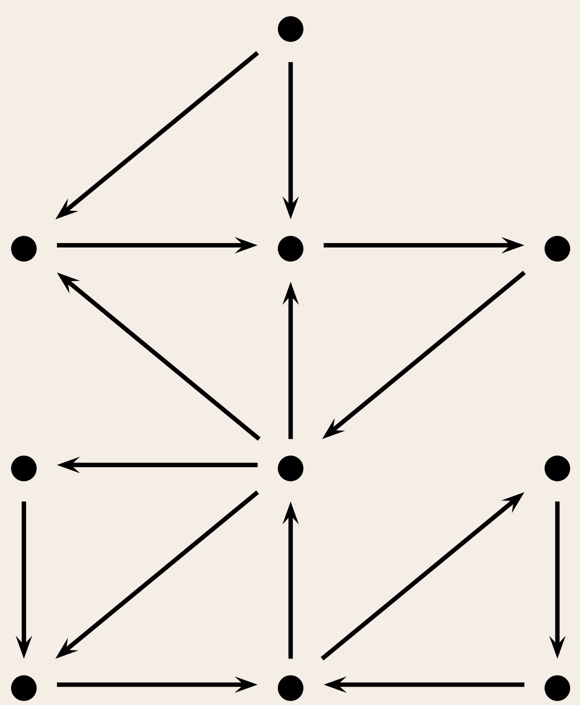
Game equivalence

Proposition. $EAE^{\omega}(G, W) \equiv (EA)^k E^{\omega}(G, W)$

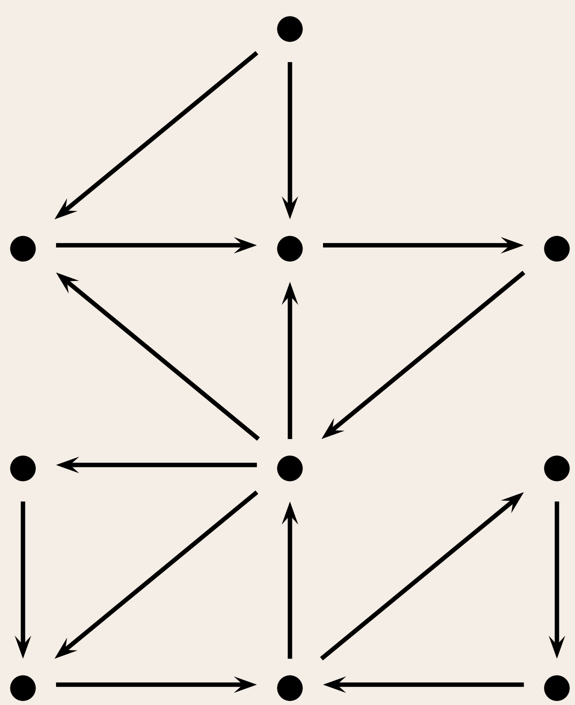
Proof. Ego wins $EAE^{\omega}(G, W) \implies$ Ego wins $(EA)^k E^{\omega}(G, W)$.

Let $k = 2$.

$EAE^{\omega}(G, W)$



$(EA)^2 E^{\omega}(G, W)$



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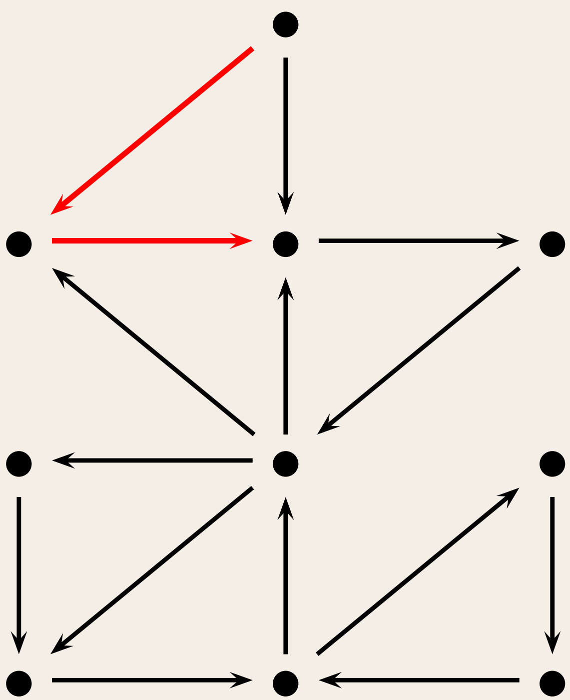

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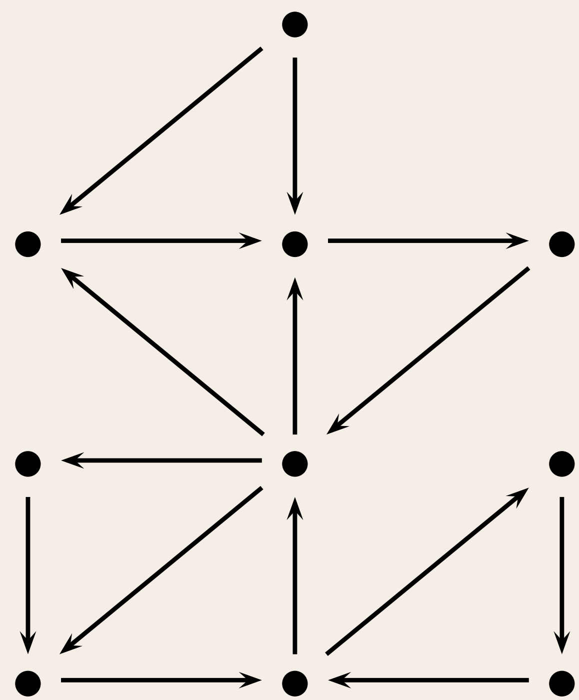

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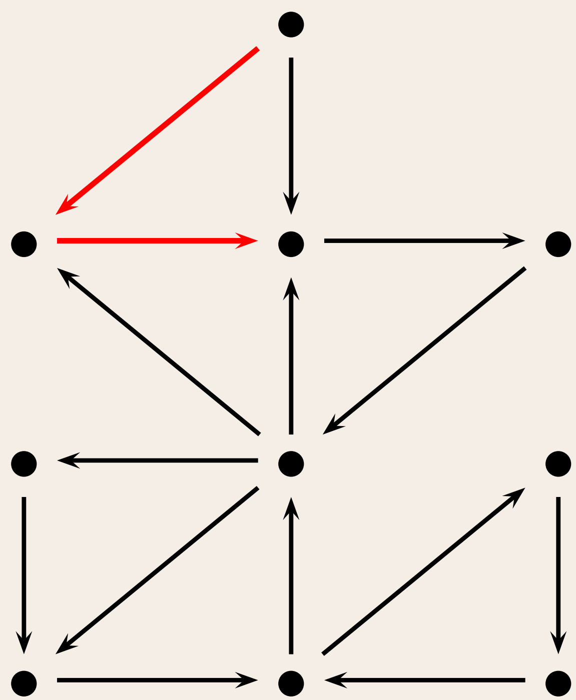

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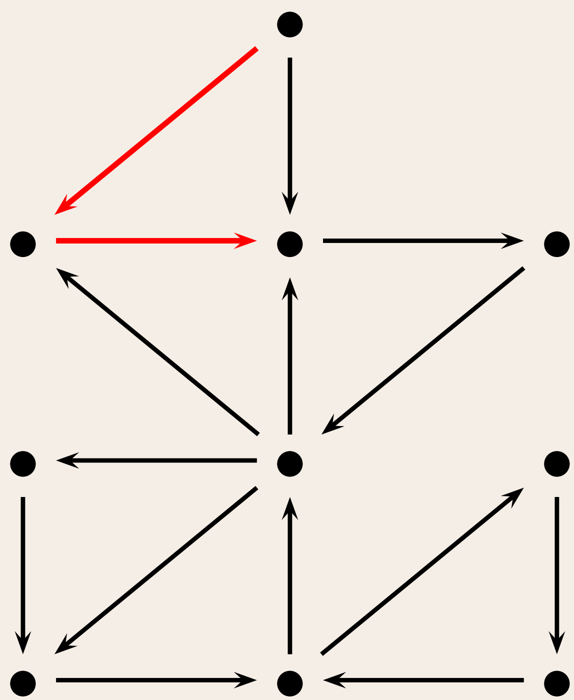

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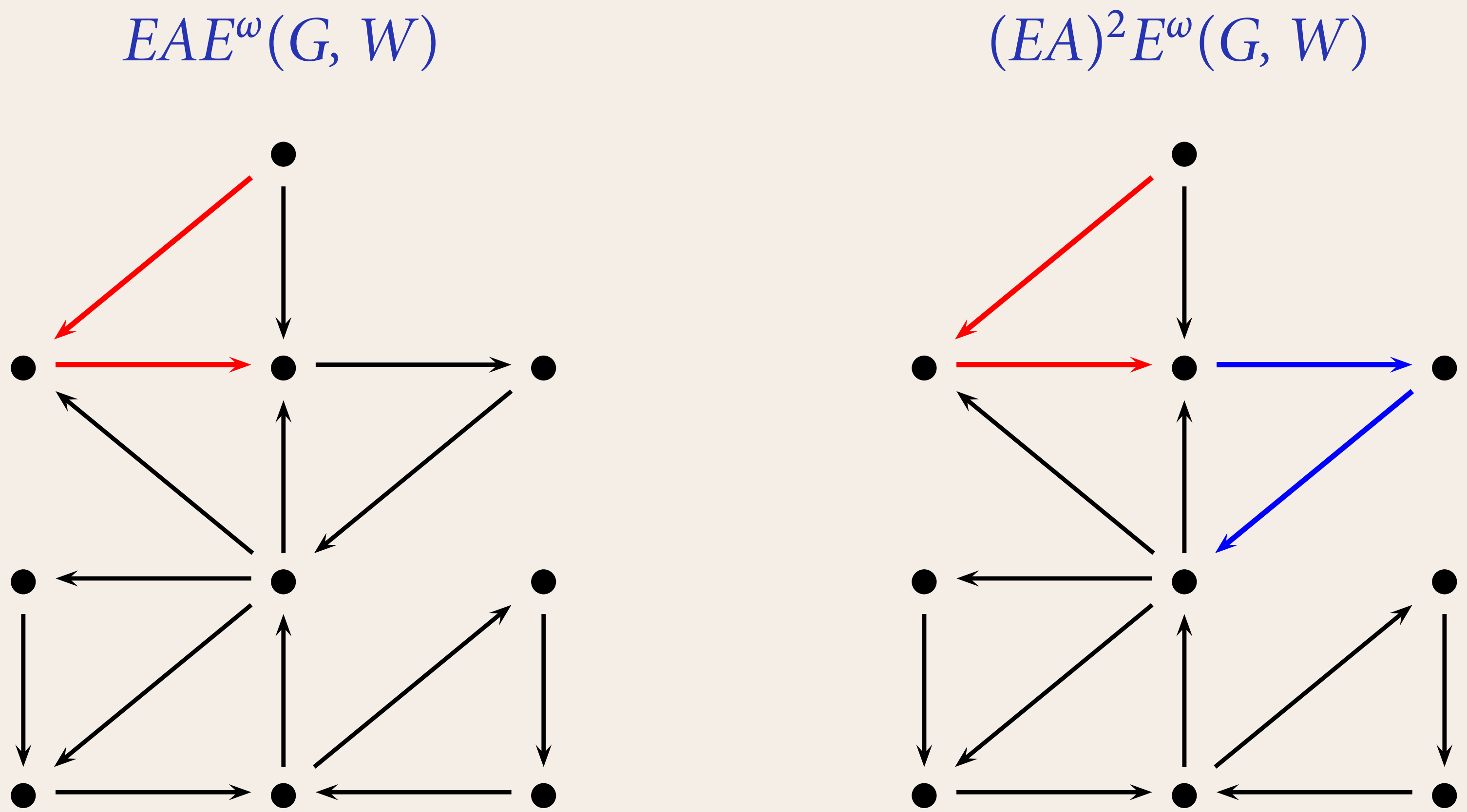

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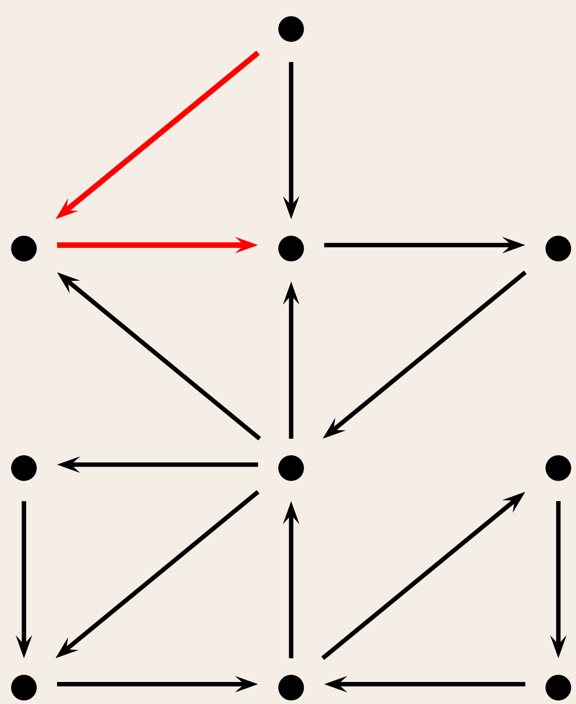

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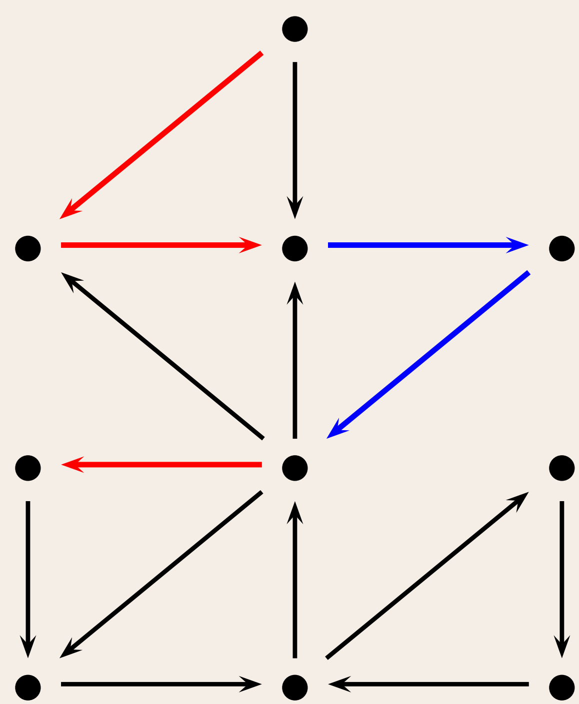

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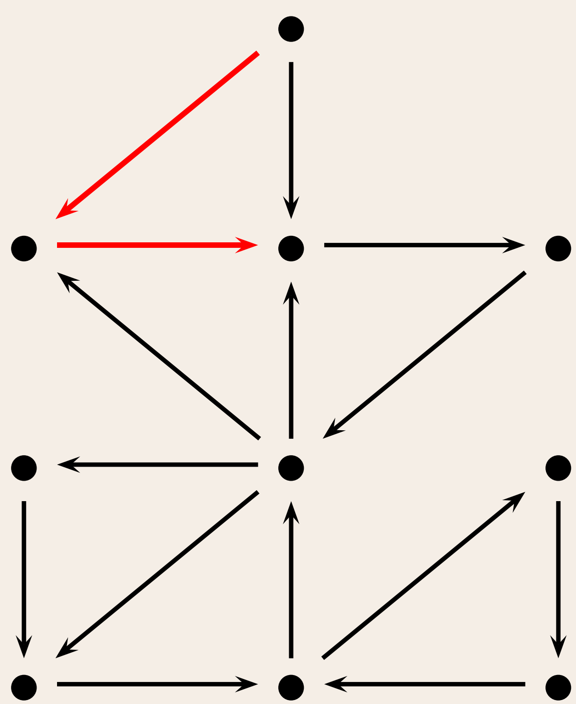

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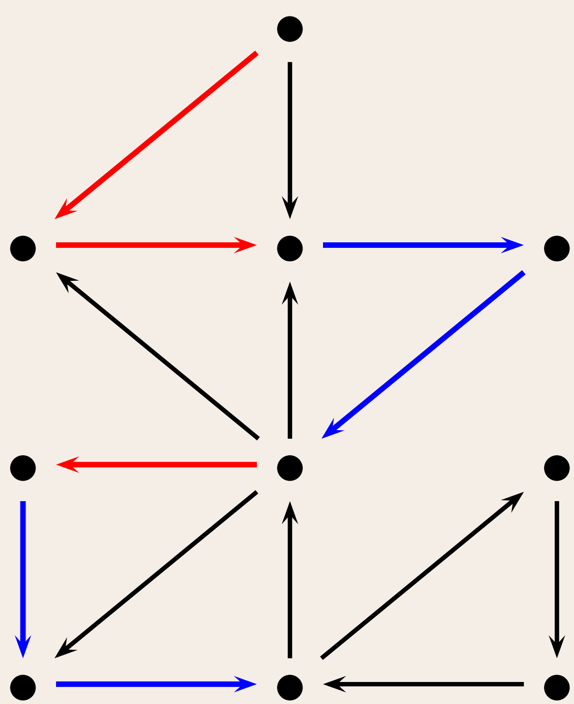

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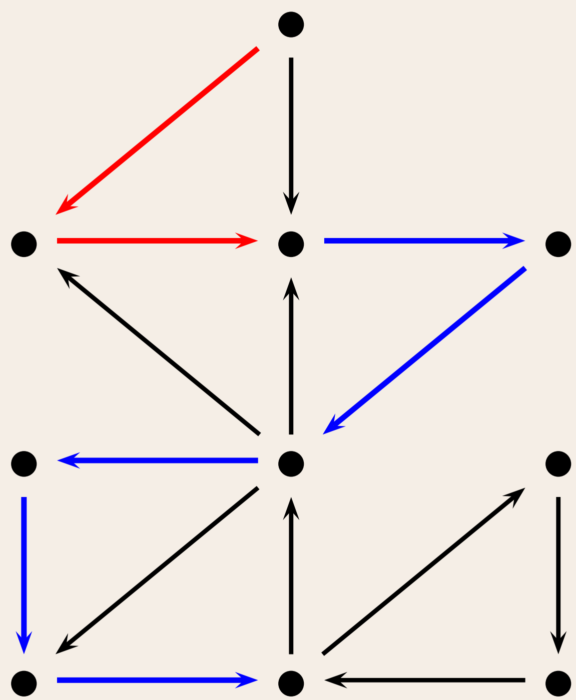

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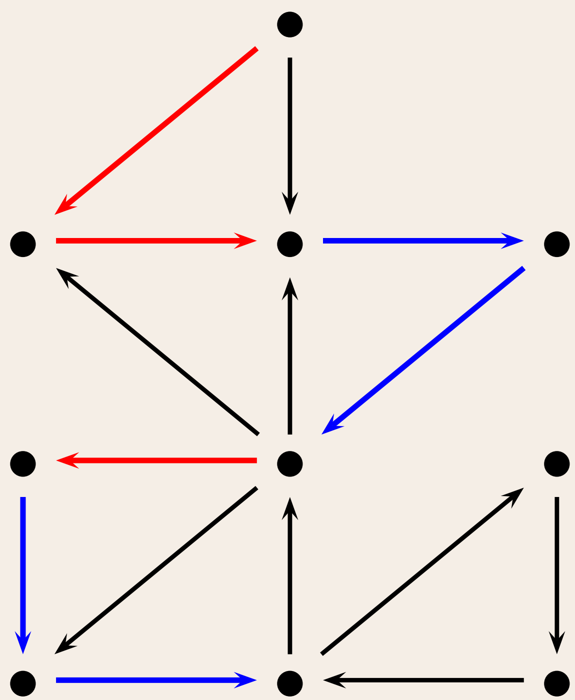

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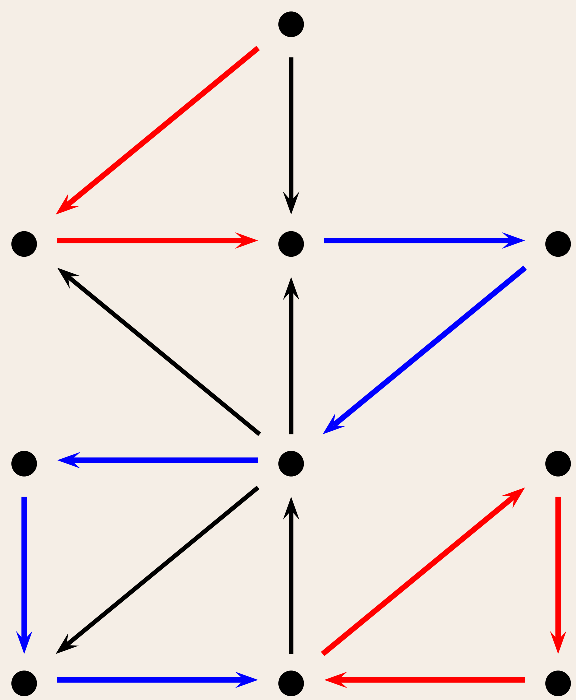

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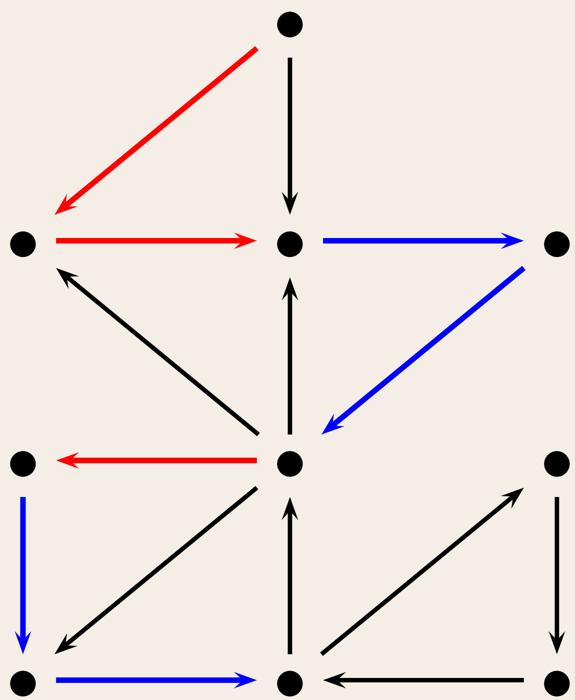

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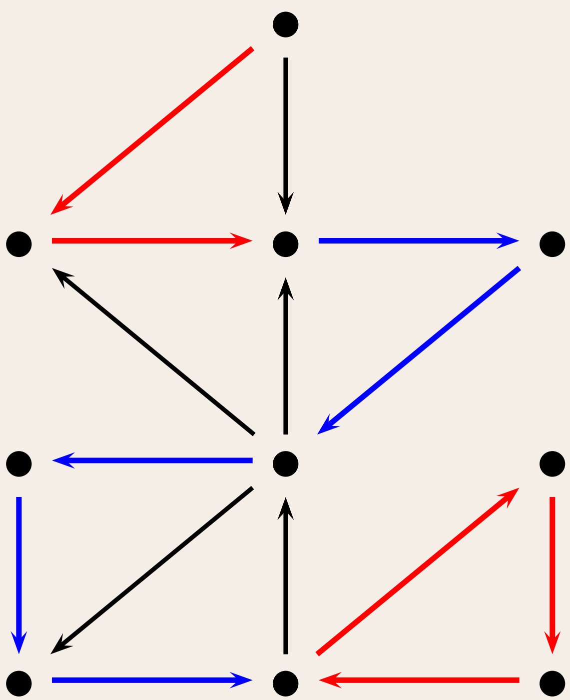

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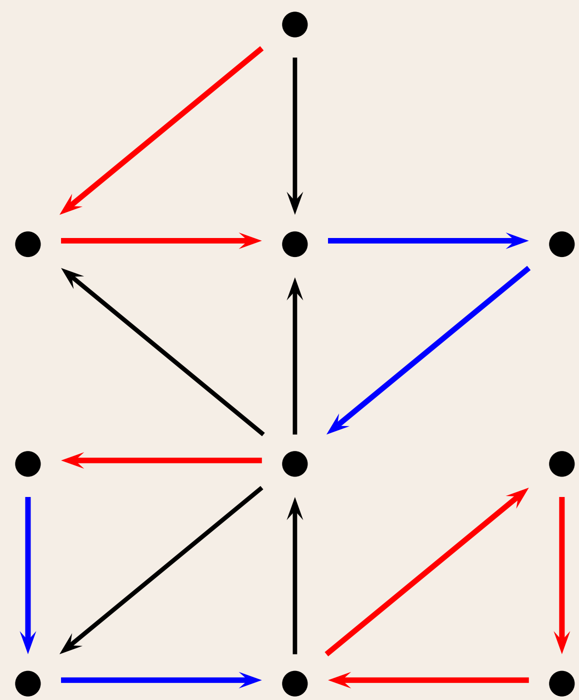

Alternations: one, two, three, infinity

The hierarchy defined by the game prefixes collapses (Pistore/Vardi)

Theorem. For any game graph G and any winning condition W

$$
\begin{array}{ccccc}
E^\omega(G,W) & \succeq & EAE^\omega(G,W) & \succeq & AE^\omega(G,W) & & \\
 & & \rotatebox{-90}{$\succeq$} & & \rotatebox{-90}{$\succeq$} & & \\
 & & (EA)^\omega(G,W) & \succeq & (AE)^\omega(G,W) & & \\
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 & & EA^\omega(G,W) & \succeq & AEA^\omega(G,W) & \succeq & A^\omega(G,W)
\end{array}
$$

Every path game over (G, W) is equivalent to one of these eight games.

Remark. This holds for games with arbitrary payoff functions (that may take other values than 0 and 1). The games need not be determined.

Alternations: one, two, three, infinity

Start conditions: *Let's have a drink first.*

$$
\begin{array}{ccccccc}
E^\omega(G,W) & \succeq & EAE^\omega(G,W) & \not\succeq & AE^\omega(G,W) & & \\
 & & \mathbin{\rotatebox[origin=c]{90}{$\succeq$}} & & \mathbin{\rotatebox[origin=c]{90}{$\succeq$}} & & \\
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$$

Won by Ego | **Won by Alter**

Alternations: one, two, three, infinity

Reachability conditions: *Some day, we'll have a drink.*

Guarantee conditions: *Every day, we have a drink.*

$$
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\end{array}
$$

Co-Büchi conditions:

Someday I will ride to the sunset and never come back.

$$E^\omega(G,W) \succeq EAE^\omega(G,W) \succeq AE^\omega(G,W)$$

Won by Ego

$$\begin{array}{ccc} EAE^\omega(G,W) & & AE^\omega(G,W) \\ \curlyvee & & \curlyvee \end{array}$$

Won by Alter

$$(EA)^\omega(G,W) \succeq (AE)^\omega(G,W)$$

$$\begin{array}{ccc} \curlyvee & & \curlyvee \end{array}$$

$$EA^\omega(G,W) \succeq AEA^\omega(G,W) \succeq A^\omega(G,W)$$

Büchi conditions: *Again and again someone will play the harmonica.*

$$
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E^\omega(G,W) & \succeq & EAE^\omega(G,W) & \succeq & AE^\omega(G,W) & & \\
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Path games in descriptive set theory

Banach-Mazur game $G^{**}(W)$:

(original version, see *"Mathematics from the Scottish Café"*): for a given winning condition $W \subseteq \mathbb{R}$, Ego first selects an interval $d_1 \subset \mathbb{R}$, then Alter chooses a subinterval $d_2 \subset d_1$, the Ego selects a further refinement $d_3 \subset d_2$, and so on ...

Ego wins, iff $\bigcap_{n\in\omega} d_n$ contains an element of W.

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Central issue in descriptive set theory: Characterise determined games by topological properties of the winning conditions.

Topology: Cantor space and Baire space

On B^ω, define a topology with basic open sets $O(x) := x \cdot B^\omega$, for $x \in B^*$.

- L is open $\iff$ $L = W \cdot B^\omega$ (for some $W \subseteq B^*$)
- L is closed $\iff$ $B^\omega - L$ is open $\iff$ $L = [T]$
  where $[T]$ is the set of infinite branches of a tree $T \subseteq B^*$.
- L is nowhere dense if the closure of L contains no non-empty open set
- L is meager if it is a countable union of nowhere dense sets.

For $B = \{0, 1\}$ this is Cantor space; for $B = \omega$ this is Baire space.

Examples:

– $O(x) = x \cdot B^\omega$ is clopen (both closed and open)

– in Cantor space, $1^*0\{0, 1\}^\omega$ is open, but not closed;
its complement $\{1^\omega\}$ is closed (and meager), but not open.

Borel sets

The class of Borel sets is the closure of the open sets under countable union and complementation.

Borel sets form a natural hierarchy of sets $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$, for $1 \leq \alpha < \omega_1$.

The first levels of the Borel hierarchy:

$\mathbf{\Sigma}^0_1$	(or G) :	the open sets
$\mathbf{\Pi}^0_1$	(or F) :	the closed sets
$\mathbf{\Sigma}^0_2$	(or F_σ) :	countable unions of closed sets
$\mathbf{\Pi}^0_2$	(or G_δ) :	countable intersections of open sets

Determinacy of Banach-Mazur games

Theorem (Banach-Mazur) In the game $(EA)^\omega(T^\omega, W)$

(1) Alter has a winning strategy $\iff$ W is meager.

(2) Ego has a winning strategy $\iff$ there exists a word $x \in \omega^*$ such that $(x \cdot \omega^\omega - W)$ is meager (W co-meager in a basic open set).

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Remark. Standard winning conditions used in applications (e.g. all winning conditions in S1S) are in low levels of the Borel hierarchy.

Planning in nondeterministic domains

Planning domain: transition system $G := (V, (E_a)_{a \in A}, (P_b)_{b \in B})$

Planning goal: property of execution paths, specified by $\varphi \in \mathrm{LTL}$

Plan: $\pi : V^* \to A$, assigns to each history an action

Execution tree: if the planning domain G is deterministic, then π defines a unique execution path. However, if G is nondeterministic, an action may have several outcomes, and a plan π then has not only one execution path, but an execution tree $\mathcal{T}_{G,\pi}$.

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It may be unrealistic to expect that all execution paths of a plan π satisfy the goal φ. On the other side, it is too optimistic to assume that a plan is good if just one execution path is consistent with φ.

Path games for planning

Pistore/Vardi study nondeterministic planning by means of path games: a plan π is good for the goal φ on domain G if Ego wins an associated game on the execution tree $\mathcal{T}_{G,\pi}$.

- **Weak planning**
  There is a path in $\mathcal{T}_{G,\pi}$ that satisfies φ.
- **Strong planning**
  Every path in $\mathcal{T}_{G,\pi}$ satisfies φ.
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  Every partial execution of π can be extended to a successful path.

Path games for planning

Pistore/Vardi study nondeterministic planning by means of path games: a plan π is good for the goal φ on domain G if Ego wins an associated game on the execution tree $\mathcal{T}_{G,\pi}$.

- **Weak planning** $E^\omega(\mathcal{T}_{G,\pi}, \varphi)$
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Theorem. The planning problem for LTL-goals, described by path games, can be solved by automata-based methods and is 2EXPTIME-complete.

Positional determinacy

A strategy is positional or memoryless if it only depends on the current position, and not on the history of the play.

Proposition. If $W \in \mathbf{\Sigma}_2^0$ (countable union of closed sets), and Ego has winning strategy for the game $(EA)^\omega(G, W)$, then he also has a positional winning strategy.

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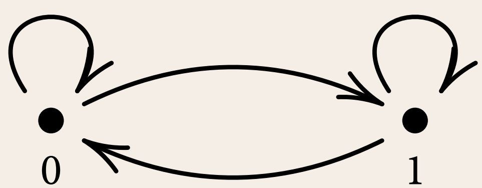
This is not always true for $W \in \Pi_2^0$:

G_2 :



$$W = \{\pi \in \{0,1\}^\omega : (\forall m)(\exists n > m) \, |\{i < n : \pi(i) = 0\}| \geq n/2\}$$

(infinitely many initial segments of π have more zeros than ones)

Ego has a winning strategy for $EA^\omega(G_2, W)$, but no positional one.

Muller, parity, and S1S winning conditions

Game graph $\mathcal{G} = (V, E)$ with colouring $\lambda : V \to \{0, \ldots, d-1\}$.

Logical winning conditions: given by formula φ in some logic on infinite paths, with predicates $\lambda(v) = i$ ($i < d$), such as

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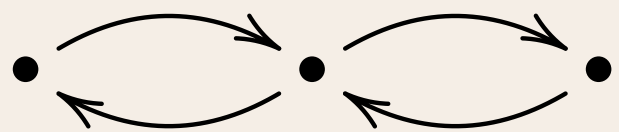
Parity condition: Ego wins if the least colour seen infinitely often is even.

Positional determinacy of classical graph games

For the more common, single step games on graphs

- parity games are positionally determined
- Positional strategies do not suffice for Muller games.

Example:

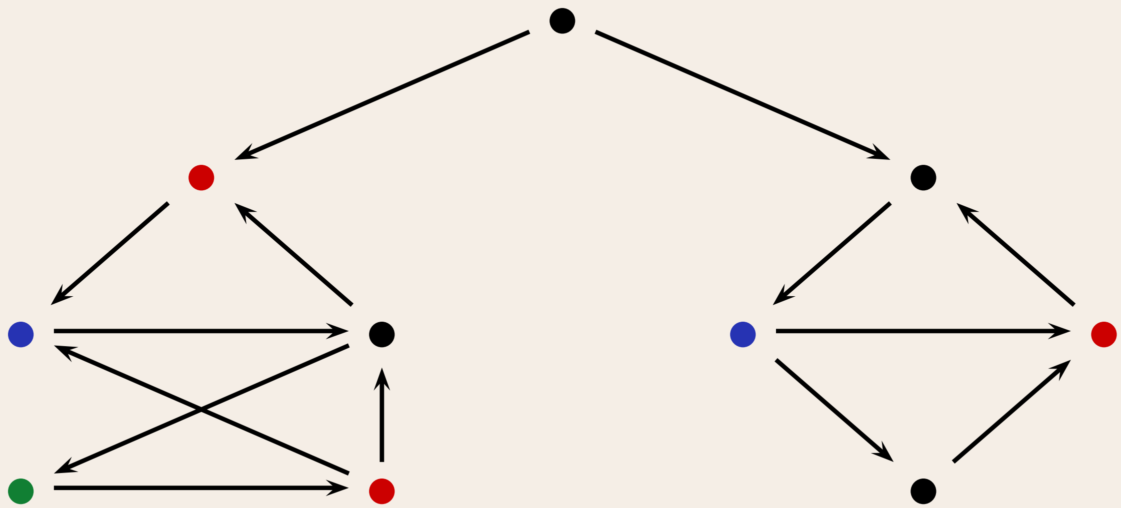


winning condition: all positions must occur infinitely often

Positional Determinacy of Muller Path Games

Proposition. Muller path games $(EA)^{\omega}(G, \mathcal{F})$ are positionally determined.

Proof. Decompose G into its strongly connected components



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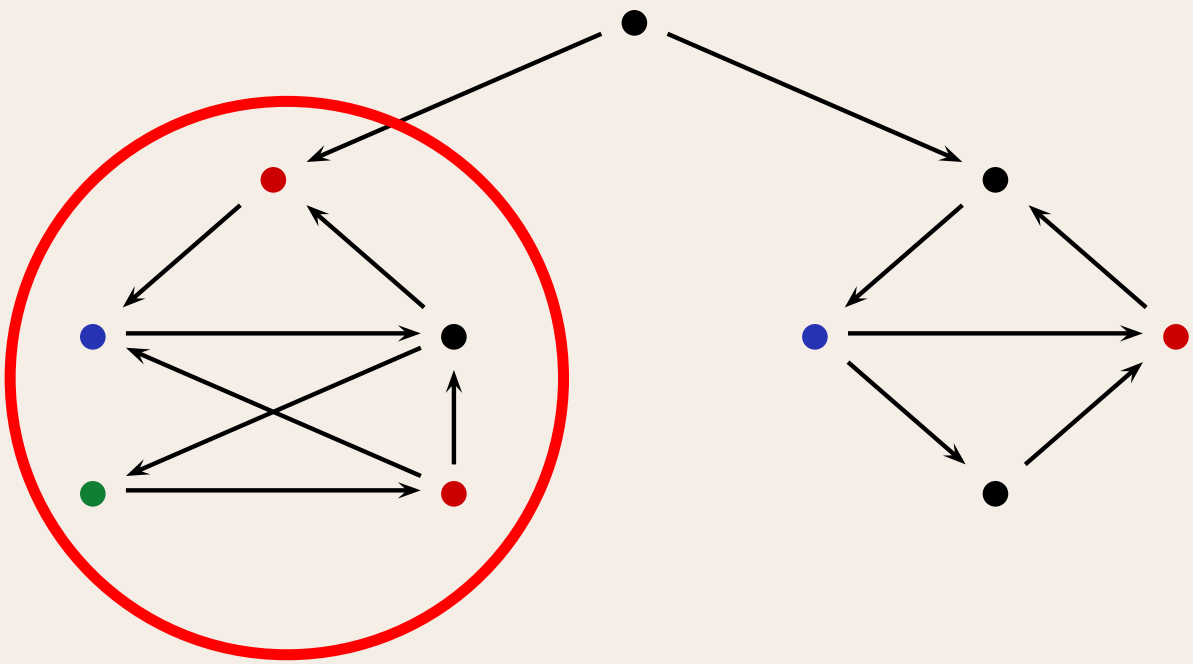

Ego wins iff there is a leaf component C such that $\lambda(C) \in \mathcal{F}$.

But then he also wins with a positional strategy.

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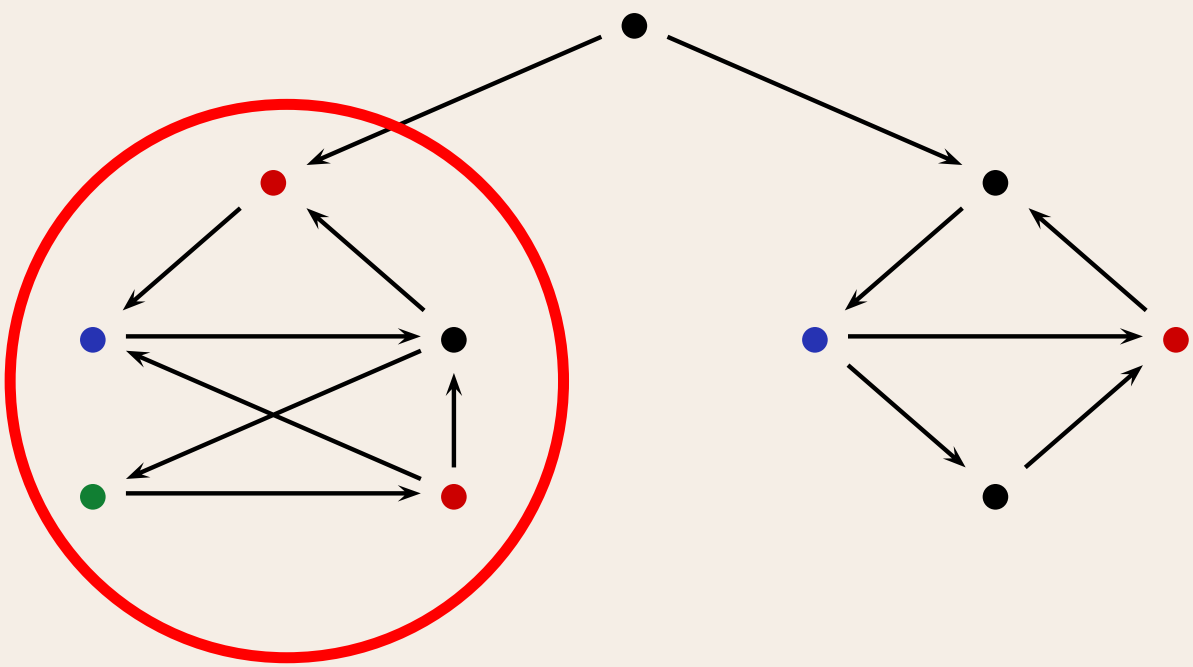

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Corollary. Muller path games can be solved in time $O(|G| \cdot |\mathcal{F}|)$.

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- For future conditions $\varphi \in \text{S1S}$, all path games $\gamma(G, \varphi)$ are positionally determined.

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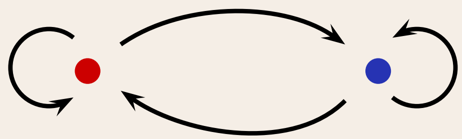
  future conditions: invariant under changes of finite initial segments
- For any game prefix γ with finite alternations there exist games $\gamma(G, \varphi)$ with $\varphi \in \text{S1S}$ that do not admit positional winning strategies.

Games that are not positionally determined

Example. φ := "the number of red nodes is odd (and finite)"



For $\gamma \in \{E^\omega, AE^\omega, EAE^\omega\}$, the game $\gamma(G, \varphi)$ (starting from the blue node) is not positionally determined.

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Question: Expressive power of γ.LTL and γ.S1S, compared to common logics on (game) graphs, like μ-calculus, CTL*, FO, and MSO ?

Definability theorem for path games

Theorem. For any game prefix γ

(1) $\gamma\,.\,\mathrm{S1S} \leq L_\mu$

(2) $\gamma\,.\,\mathrm{LTL} \equiv \gamma\,.\,\mathrm{FO} \leq \mathrm{CTL}^*$

That is, the winner of any path game with S1S resp. LTL winning condition is definable in the modal μ-calculus resp. CTL^*.

Simplification via bisimulation invariance

It suffices to prove, that on trees,

(1) $\gamma . \mathrm{S1S} \leq \mathrm{MSO}$

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MPL is monadic path logic: MSO on trees with second-order quantification restricted to (finite or infinite) paths.

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- winning a (path) game is invariant under bisimulation
- bisimulation-invariant MSO $\equiv$ L_μ (Janin/Walukiewicz)
- bisimulation-invariant MPL $\equiv$ CTL^* (Hafer/Thomas) (Moller/Rabinovitch)

The simple case: games with finite alternations

Ego wins $EAE^\omega(\mathcal{T}, \varphi) \quad \Longleftrightarrow \quad \mathcal{T} \models \psi$, where

$$\psi := (\exists X \,.\, X \text{ finite path})(\forall Y \,.\, X \subseteq Y \land Y \text{ finite path})$$
$$(\exists Z \,.\, Y \subseteq Z \land Z \text{ infinite path} \land \varphi|_Z)$$

$\varphi|_Z := \varphi$ relativized to the path Z

- $\varphi \in \text{S1S} \quad \Longrightarrow \quad \psi \in \text{MSO}$
- $\varphi \in \text{LTL} \quad \Longrightarrow \quad \psi \in \text{MPL}$

The slightly harder case: games with infinite alternations

Winning strategy of Ego for $(EA)^\omega(\mathcal{T}, \varphi)$ on tree $\mathcal{T} = (V, E)$:
described by set $X \subseteq V$ with

- X is non-empty
- for all $x \in X$ and $y > x$ there is a $z > y$ with $z \in X$
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Obviously, this can be formalised in MSO, if $\varphi \in$ S1S.

For $\varphi \in$ FO, we have to formalise in MPL.
In fact we can even formalise in FO !

Normal form for FO:

on infinite paths, every first-order formula is equivalent to

$$\bigvee_i \Big(\exists x (\forall y \geq x) \varphi_i \wedge \forall x (\exists y \geq x) \vartheta_i \Big)$$

where φ_i and ϑ_i contain only bounded quantifiers $(Qz \leq y)$.

In terms of LTL: Every LTL-formula is equivalent to a disjunction of formulae $(\mathrm{FG}\varphi \wedge \mathrm{GF}\vartheta)$, where φ and ϑ are past-formulae.

Use this to show that on trees,

$$(EA)^\omega.\,\mathrm{FO} \leq \mathrm{FO} \quad \text{and} \quad (AE)^\omega.\,\mathrm{FO} \leq \mathrm{FO}.$$

Conclusion

- Path games are a natural kind of games on graphs.
- They arose in the wild west, but also in descriptive set theory and for planning in non-deterministic domains.
- The hierarchy defined by game prefixes collapses to eight different games.
- Many path games with S1S winning conditions (in particular all Muller games) are positionally determined.
- The winner of LTL path games can be defined in CTL*.

The End