

Countable ω -automatic Structures

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Introduction

Countable ω -automatic Structures

ω -Semigroups

Conclusions and Future Work

ω -Automatic Relations

Relation $R(w_1, \dots, w_k)$ over ω -words is **ω -automatic** if it is recognised by a **synchronous** Büchi (Parity, Muller) automaton.

Such automaton works in fact over **k -tuples** of letters, i.e. over **convolution** of the words w_1, \dots, w_k .

$$w_1 \otimes \dots \otimes w_k = \begin{bmatrix} w_1[0] \\ \vdots \\ w_k[0] \end{bmatrix} \begin{bmatrix} w_1[1] \\ \vdots \\ w_k[1] \end{bmatrix} \begin{bmatrix} w_1[2] \\ \vdots \\ w_k[2] \end{bmatrix} \dots$$

ω -Automatic Structures

For a relational structure $\mathfrak{A} = (A, \{R_i\}_k)$ a tuple of automata $\mathfrak{d} = (\mathcal{A}, \mathcal{A}_\varepsilon, \{\mathcal{A}_i\}_k)$ together with a **surjective naming function** $f : L(\mathcal{A}) \rightarrow A$ constitutes an **ω -automatic presentation** of \mathfrak{A} when:

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- ▶ every $L(\mathcal{A}_i)$ has the same arity as R_i ,
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The presentation is said to be **injective** whenever f is, in which case \mathcal{A}_ε **can be omitted**.

Interesting ω -Automatic Structures

Boolean Algebra $(\mathcal{P}(\mathbb{N}), \cup, \cap, {}^c, \emptyset, \mathbb{N})$

- ▶ $A \subseteq \mathbb{N} \rightsquigarrow w_A : w_A[i] = 1 \iff i \in A$
- ▶ $\cup \rightsquigarrow \max$
- ▶ $\cap \rightsquigarrow \min$
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Atomless Boolean Algebra $(\mathcal{P}(\mathbb{N}), \cup, \cap, {}^c, \emptyset, \mathbb{N}) / \sim$

$$A \sim B \iff |A \Delta B| < \omega$$

- ▶ the relation \sim is ω -automatic (finitely many different positions)
- ▶ the **countable** atomless boolean algebra is **not automatic**
- ▶ is there an **injective presentation** of the atomless boolean algebra?

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Questions:

- ▶ Are all countable ω -automatic structures automatic?
- ▶ Are all **injectively presentable** countable ω -automatic structures automatic?
- ▶ Do all ω -automatic structures have an injective presentation?
- ▶ Do all **countable** ω -automatic structures have an injective presentation?

Injective Countable Structures are Automatic

Theorem (Blumensath, 1999)

Let \mathfrak{d}, f be an injective ω -automatic presentation of a countable structure \mathcal{A} . Then, an automatic presentation \mathfrak{d}', f' of \mathcal{A} can be constructed.

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Proposition

An ω -regular set L is countably infinite if and only if it can be written as a finite union of sets of the form $U_k \cdot w_k^\omega$ with U_k a regular set of finite words and w_k finite.

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Note: **countability of the domain of f is enough.**

Equivalence Relations with Countably many Classes

Claim: for **every** presentation (∂, f) of a countable structure \mathcal{A} it holds that in every equivalence class of ε there is an ultimately periodic word with **bounded period**.

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 $\{wu^\omega : |u| < K\}$ **and thus an automatic one.**

Example. Let $\Sigma = \{a, b, c\}$ and $\text{Inf}(w)$ be the set of letters occurring infinitely often in w .

$$E = \{(u, w) : \text{Inf}(u) = \text{Inf}(w) \text{ and } u[i] = w[i] \text{ when } u[i] \notin \text{Inf}(u) \text{ and } w[i] \notin \text{Inf}(w)\}.$$

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What are ω -Semigroups

An ω -semigroup $S = (S_f, S_\omega, \cdot, *, \pi)$ is a two-sorted algebra, where

- ▶ (S_f, \cdot) is a **semigroup**,
- ▶ $*$: $S_f \times S_\omega \mapsto S_\omega$ is the **mixed product**

$$s \cdot (t * \alpha) = (s \cdot t) * \alpha$$

- ▶ $\pi : S_f^\omega \mapsto S_\omega$ is the **infinite product** satisfying

$$s_0 \cdot \pi(s_1, s_2, \dots) = \pi(s_0, s_1, s_2, \dots)$$

as well as the **associativity rule**

$$\pi(s_0, s_1, s_2, \dots) = \pi(s_0 s_1 \cdots s_{k_1}, s_{k_1+1} s_{k_1+2} \cdots s_{k_2}, \dots).$$

Automata and Semigroups

There is a well known fundamental correspondence between recognizability by finite automata and by finite semigroups.

Morphisms of ω -semigroups preserve all three products as expected.

An ω -semigroup $S = (S_f, S_\omega)$ **recognises** a language $L \subseteq \Sigma^\omega$ via a morphism $\phi : (\Sigma^+, \Sigma^\omega) \rightarrow (S_f, S_\omega)$ if $\phi^{-1}(\phi(L)) = L$.

A language is recognised by a finite semigroup if and only if it is recognised by a non-deterministic Büchi automata.

Basic Properties of Finite Semigroups

Idempotents are elements $e \in S_f$ such that $ee = e$.

- ▶ **bounded-length representants:** there is a number M so that every $u \in \Sigma^+$ has a $v \in \Sigma^+$, $|v| < M$ with $\phi(u) = \phi(v)$
- ▶ **exponent of semigroup:** there is a number K so that for every word u the element $\phi(u^K)$ is an idempotent

Main Lemma

Lemma: Let E be an equivalence relation over $(\Sigma^2)^\omega$ recognised by a **finite ω -semigroup S** via ϕ . If E has countably many equivalence classes then for every $w, u, v \in \Sigma^*$ there is a k so that

$$\phi\begin{pmatrix} u \\ u \end{pmatrix} = \phi\begin{pmatrix} v \\ v \end{pmatrix} \implies ((wv^k)u^\omega, wv^\omega) \in E.$$

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- ▶ $\varphi(x) = \forall y By \rightarrow \neg x \in y$ defines all words that do not have a representant with bounded period

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- ▶ the set **B** of words with M -bounded period is regular
- ▶ $\varphi(x) = \forall y By \rightarrow \neg x \in y$ defines all words that do not have a representant with bounded period
- ▶ let $x = wv^\omega$ be an ultimately periodic word satisfying $\varphi(x)$
- ▶ take shorter u with $\begin{pmatrix} u \\ u \end{pmatrix} = \begin{pmatrix} v \\ v \end{pmatrix}$, contradict lemma

Algebraic Lemma

Let T be a relation over $(\Sigma^2)^\omega$ recognised by a finite semigroup S via ϕ . If there exist words $u, v \in \Sigma^+$ for which $\binom{u}{u} = \binom{v}{v}$ and $u^\omega \neq v^\omega$, then words $a \neq b$ can be found satisfying:

1. $\binom{a}{a}$ and $\binom{b}{b}$ are idempotent,
2. $\binom{a}{a} = \binom{b}{b}$,
3. $|a| = |b|$,
4. $\binom{a}{b} \binom{a}{a} = \binom{a}{b} \binom{b}{b} = \binom{a}{b}$,
5. $\binom{b}{a} \binom{a}{a} = \binom{b}{a} \binom{b}{b} = \binom{b}{a}$,

If T is transitive and for all k $((wv^k)u^\omega, wv^\omega) \notin T$ then there is a w' :

6. $\binom{w'}{w'} \binom{a}{a} = \binom{w'}{w'} \binom{b}{b} = \binom{w'}{w'}$,
7. $(w'a^\omega, w'b^\omega) \notin T$.

Proving the Algebraic Lemma

Take $\hat{u} = (u^K)^{|v|}$, $\hat{v} = (v^K)^{|u|}$ and put

$$a = \hat{u}\hat{u}, \quad b = \hat{v}\hat{v}, \quad w' = w\hat{v}.$$

We prove (7): $(w'a^\omega, w'b^\omega) \notin T$ if T is transitive and $(w'u^\omega, w'v^\omega) \notin T$.

- ▶ assume to the contrary that $(w'a^\omega, w'b^\omega) \in T$
- ▶ calculate $\binom{w'a^\omega}{w'b^\omega} =$

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$$= \binom{w'(\hat{u}\hat{u})^\omega}{w'(\hat{v}\hat{u})^\omega} =$$

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► calculate $\binom{w'a^\omega}{w'b^\omega} = \binom{\hat{v}}{\hat{v}} = \binom{\hat{v}}{\hat{v}} \binom{\hat{v}}{\hat{v}}$

$$= \pi\left(\binom{w}{w}, \binom{\hat{v}}{\hat{v}}, \binom{\hat{u}}{\hat{v}}, \binom{\hat{u}}{\hat{u}}, \binom{\hat{u}}{\hat{v}}, \binom{\hat{u}}{\hat{u}}, \dots\right) =$$

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$$= \pi\left(\begin{pmatrix} w' \\ w' \end{pmatrix}, \begin{pmatrix} \hat{v} \\ \hat{v} \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{u} \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{u} \end{pmatrix}, \dots\right) =$$

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- ▶ **by transitivity** $(w'(\hat{u}\hat{u})^\omega, w'(\hat{v}\hat{v})^\omega) \in T$, but $(w'u^\omega, w'v^\omega) \notin T$. \downarrow

Proving the Main Lemma (1)

Start with a relation E recognised by S via ϕ and (by contradiction) the words w, u, v so that $\binom{u}{u} = \binom{v}{v}$ and $((wv^k)u^\omega, wv^\omega) \notin E$.

Take w', a, b as before: $w'(ab)^\omega$ and $w'(ba)^\omega$ are not equivalent.

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- ▶ assume $(w'(ab)^\omega, w'(ba)^\omega) \in E$
- ▶ calculate $\binom{w'(ba)^\omega}{w'(abaa)^\omega} = \binom{w'(ba)^\omega}{w'(ab)^\omega}$.
- ▶ so **by symmetry** $(w'(ba)^\omega, w'(abaa)^\omega) \in E$
- ▶ and **by transitivity** $(w'(ab)^\omega, w'(abaa)^\omega) \in E$
- ▶ calculate $\binom{w'(abab)^\omega}{w'(abaa)^\omega} =$

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Start with a relation E recognised by S via ϕ and (by contradiction) the words w, u, v so that $\binom{u}{v} = \binom{v}{u}$ and $((wv^k)u^\omega, wv^\omega) \notin E$.

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 $= \pi\left(\binom{w'}{w'}, \binom{a}{a}, \binom{b}{b}, \binom{a}{a}, \binom{b}{a}, \binom{a}{a}, \binom{b}{b}, \binom{a}{a}, \binom{b}{a} \dots\right) =$

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- ▶ but by the algebraic lemma $(w'b^\omega, w'a^\omega) \notin E$

Proving the Main Lemma (2)

Knowing that $(w'(ab)^\omega, w'(ba)^\omega) \notin E$, let us take any two words x_1, x_2 of the form $w'(ab, ba)^\omega$ that differ on infinitely many positions. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$

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$$= \begin{pmatrix} w' \\ w' \end{pmatrix} \begin{pmatrix} w_0 \\ w_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} \begin{pmatrix} w_1 \\ w_1 \end{pmatrix} \cdots \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} \begin{pmatrix} w_n \\ w_n \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} w_{n+1} \\ w_{n+1} \end{pmatrix} \cdots =$$

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Thus no two such words are equivalent, which gives an **uncountable set of non-equivalent words**.

Overview

Introduction

Countable ω -automatic Structures

ω -Semigroups

Conclusions and Future Work

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- ▶ and have an **injective presentation**
- ▶ there is a way to transfer **transitivity to algebraic structure**

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Open Problems:

- ▶ understand better **how transitivity is reflected** in the algebraic structure of the underlying ω -semigroup
- ▶ do all ω -automatic structures have an **injective presentation**?
 - ▶ show that **atomless boolean algebra** does not have one
- ▶ are all countable ω -**tree-automatic** structures tree-automatic?
- ▶ do **infinity quantifiers** preserve regularity in general?

Thank You