Game reductions and FAR for infinite games

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Motivation I

- Parity games with finite number of colours are positionally determined
- Muller games with finite number of colours are not positionally determined
- Finitely colored Muller games can be reduced to parity games using *latest appearance record* (LAR) or *Zielonka tree* and associated memory

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Motivation II

- Infinitely (ω) coloured min-parity games are positionally determined
- Moreover, for every ω-coloured positionally determined Muller game there is an isomorphic min-parity winning condition
- ω-coloured max-parity games are not positionally determined
- There are simple examples of ω-coloured Muller games that do not have a finite memory strategy.

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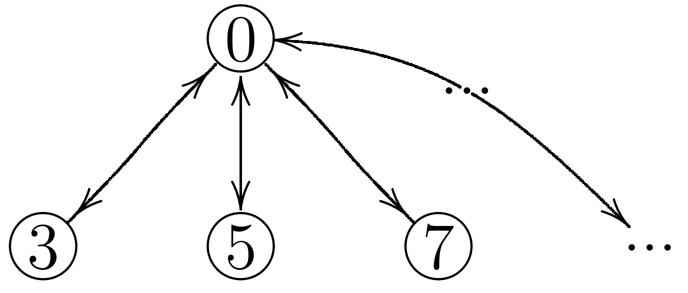
Goals

- Consider the following max-parity game:



- 0 wins with an intuitively simple strategy
- No finite memory strategy
- Goals:
  - Find a way to analyse different kinds of memory in games
  - Define a suitable memory for infinite Muller games

Game Reductions with Memory

Adding Memory to Games

- *Memory* is given by a set of states M with an selected initial state m_0 and an update function

$$\mu \ : \ V \times M \to M$$

- We will extend the game arena

$$G = (V_0 \sqcup V_1, E) \rightsquigarrow (V_0 \times M \sqcup V_1 \times M, E') = G'$$

- Moves

$$v \to w \rightsquigarrow (v, m) \to (w, \mu(v, m))$$

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Reducing Games

- Extending a play

$$\pi = v_0 \rightarrow v_1 \rightarrow \ldots \rightsquigarrow$$

$$\rightsquigarrow (v_0, m_0) \rightarrow (v_1, \mu(v_0, m_0)) \rightarrow \ldots = \pi'$$

- G reduces with memory $\mathcal{M} = (M, m_0, \mu)$ to G' if there is a winning condition for G' such that

$$0\ wins\ \pi \text{ in } G \iff 0\ wins\ \pi' \text{ in } G'$$

- G is an $\mathcal{M}$ *strategy game* if it reduces with $\mathcal{M}$ to a positionally determined game

Using Game Reductions

- *Example:* LAR reduction of Muller games to parity games
- Advantage: we capture different kinds of memories and prove only equivalences of plays
- Drawback: we capture only reductions for both players
- For example 0-positionality of Rabin games can not be proved directly with reductions

Finite Appearance Record

Finite Appearance Record

We consider ω-coloured Muller games.
(M, m_0, μ) is a *finite appearance record* if

$$M = (\omega \sqcup N)^d$$

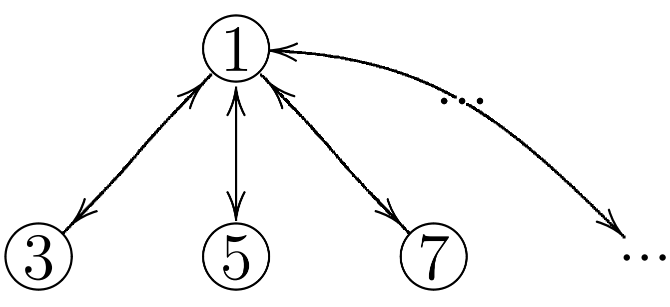
for some finite set N and number d (FAR dimension) and when

$$\mu(v, c_1, \ldots, c_d) = (k_1, \ldots, k_d)$$

then k_i is either the colour if v or $k_i \in N$ or $k_i = c_j$.
Example: LAR is a FAR if we take d to be equal to the number of colours in the finitely coloured game and $N = \{\natural, B\}$.

FAR Properties

- Resembles register machines, automata with infinite signature
- We can always increase the colour we see with 1-dimensional FAR
- We can infinitely often visit infinitely many colours with 2-dimensional FAR



Reducing Infinite Games

ω-coloured Muller Games

- We consider infinitely coloured Muller games but with finite winning condition:

$$\mathcal{F} = \{W_1, W_2, \ldots, W_N\}$$

- $W_i \subseteq \omega$ possibly infinite
- If $\mathcal{F} = \{W_1, W_2, \ldots\}$ contains infinitely many sets then we can construct a game that does not reduce to a positionally determined one with any FAR
- Since we consider only finitely many W_i's we do not capture parity games

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Muller Games Reduction

- *Theorem:* Muller games with finite winning condition can be reduced to finitely coloured Muller games using FAR
- For $\mathcal{F} = \{W_1, W_2, \ldots, W_N\}$ we are using FAR of dimension $3 \cdot N$
- Reduction idea: for each W_i we use 3 registers, two to visit all colours in W_i infinitely often and the remaining one to make sure no other colours are seen repeatedly
- *Corrolary:* Each Muller game with a finite winning condition is an FAR strategy game

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Conclusions and Open Problems

- Game reductions are a convenient method to analyse what type of memory is needed to win specific games
- Finite appearance record is enough to win ω-coloured Muller games with finite winning conditions
- For ω-coloured Muller games that have a Zielonka tree we can keep in memory the position on a branch of the tree and win with such memory
- Some Muller games with infinite winning condition can be won with FAR - which ones?
- Can max-parity games be won with FAR?

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Thank you!