Unfolding Parity Games

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RWTH Aachen

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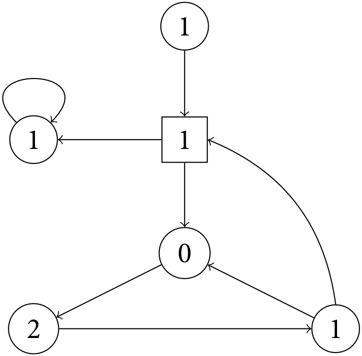
Parity Games

$\mathcal{G} = (V, V_0, V_1, E, \Omega)$ and $vE \neq \emptyset$ for all $v \in V$



Player 0 wins $\mathcal{G}$ from v_0 when she has a strategy σ so that for all strategies ρ of Player 1 the minimal colour appearing infinitely often on $\pi_{v_0}(\sigma, \rho)$ is even.

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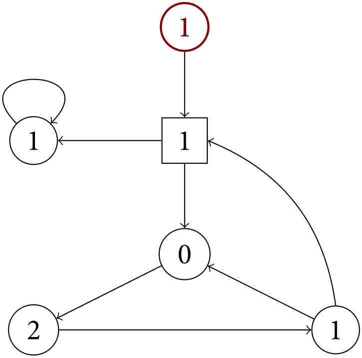

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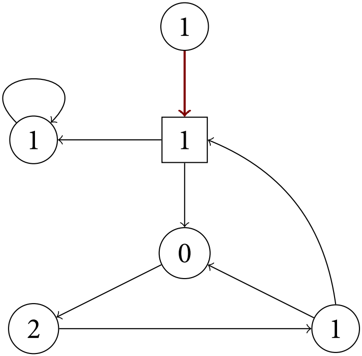

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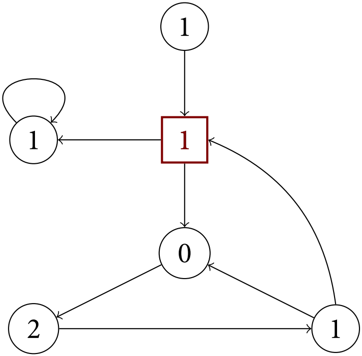

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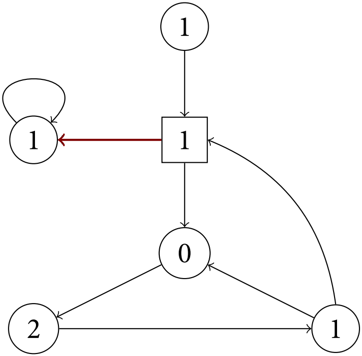

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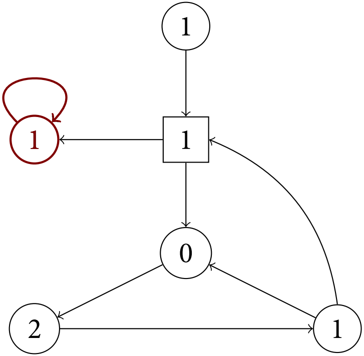

Outcome: π won by Player 1 since the lowest colour on the cycle is odd.

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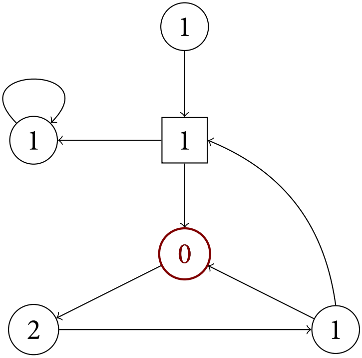

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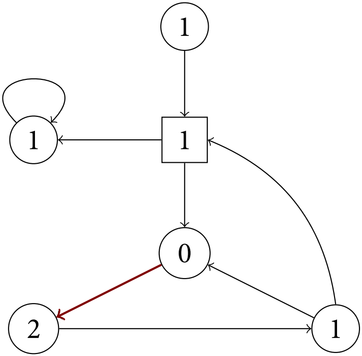

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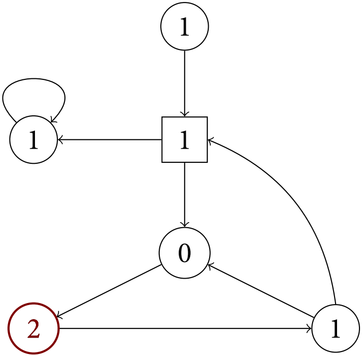

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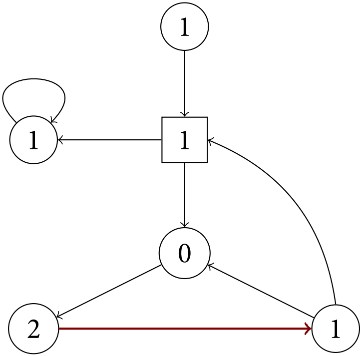

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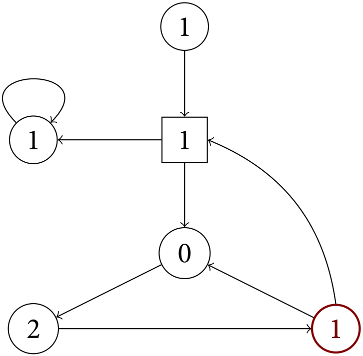

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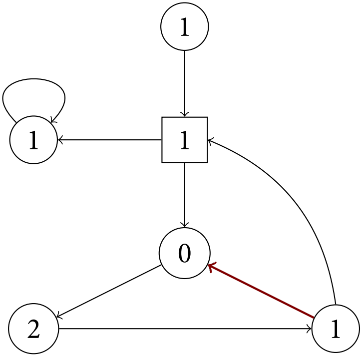

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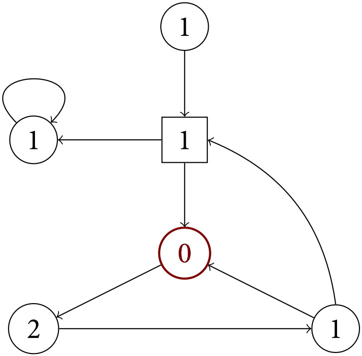

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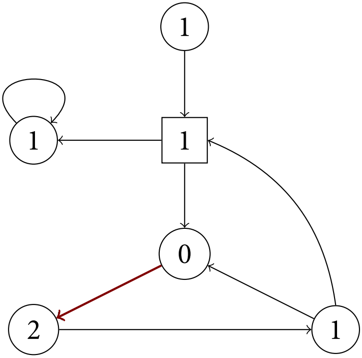

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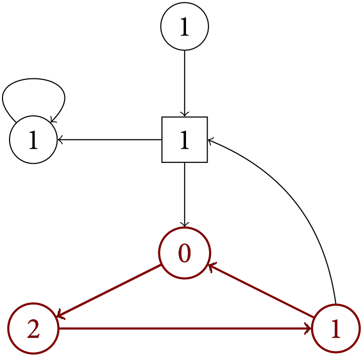
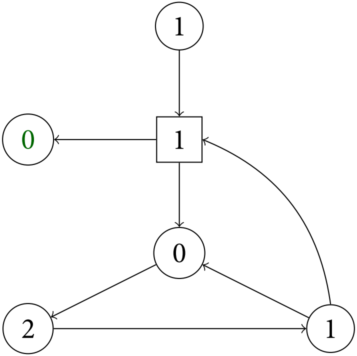


Outcome: π won by Player 0 since the lowest colour on the cycle is even.

Parity Games with End Nodes

$\mathcal{G} = (V, V_0, V_1, E, \lambda, \Omega)$



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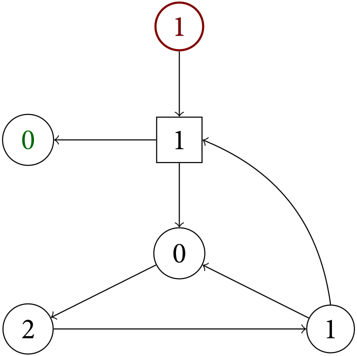

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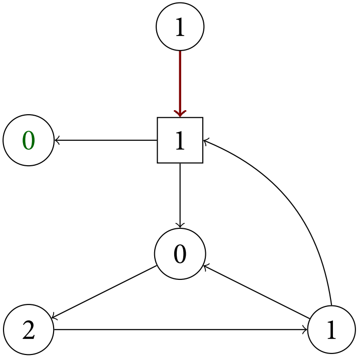

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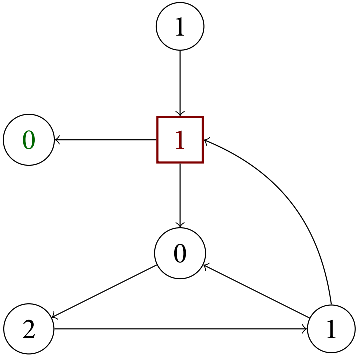

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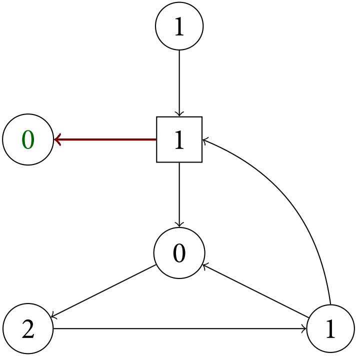

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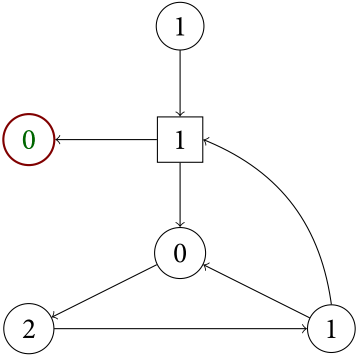

Outcome: $p(\pi) = 0$ just by **payoff** at end node (any real number).

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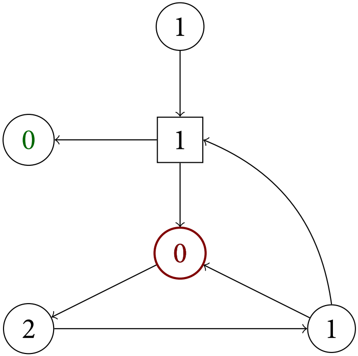

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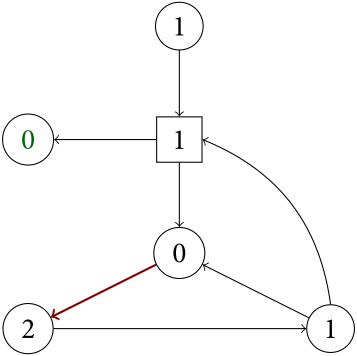

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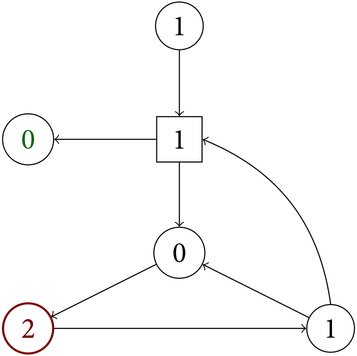

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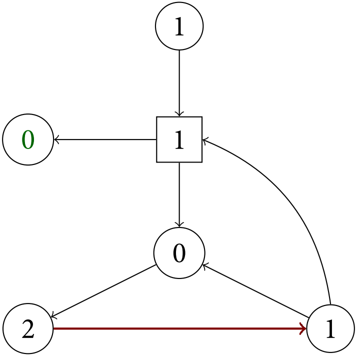

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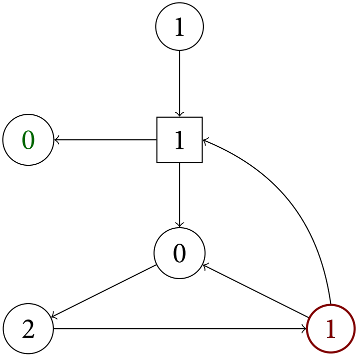

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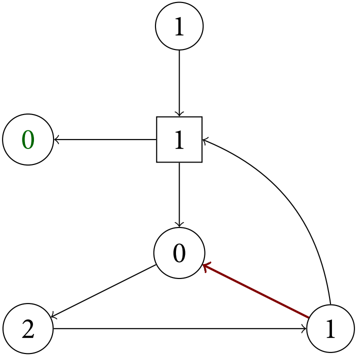

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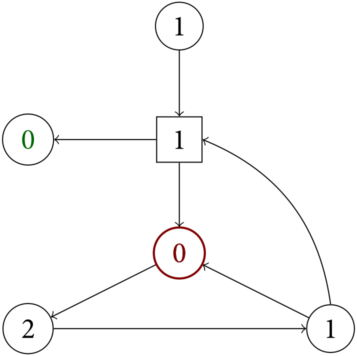

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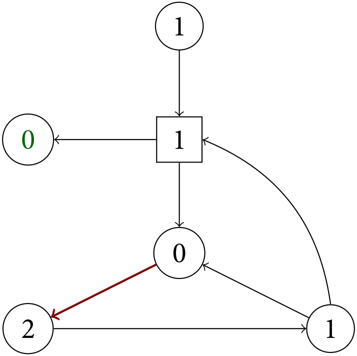

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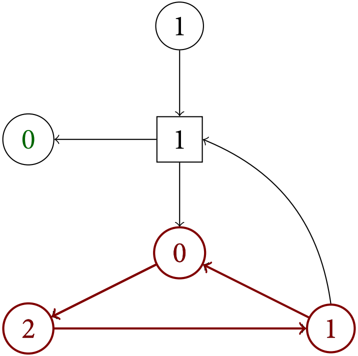

Outcome: $p(\pi) = 1$ since the lowest colour on the cycle is even.

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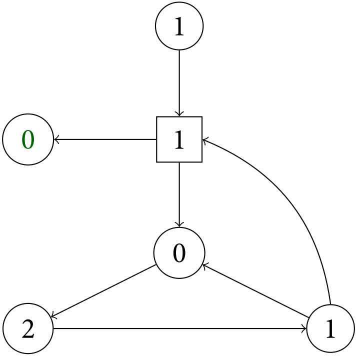
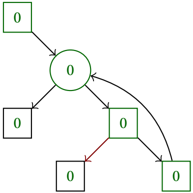


$\text{val}\mathcal{G}(v_0) = \sup_\sigma \inf_\rho p(\pi_{v_0}(\sigma, \rho)) = \inf_\rho \sup_\sigma p(\pi_{v_0}(\sigma, \rho))$ **(determinacy)**

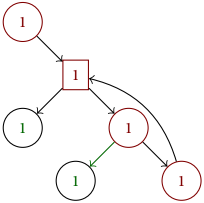
One or Two Colours

One Colour

Safety Games

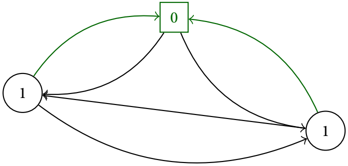


Reachability Games

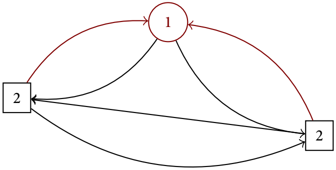


Two Colours

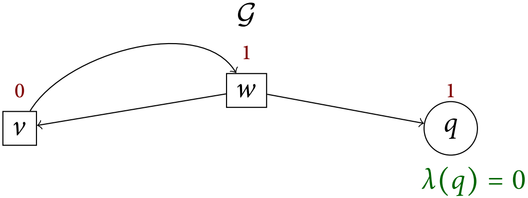
Büchi Games



Co-Büchi Games



Unfolding of a game $\mathcal{G}$: **sequence of games $\mathcal{G}_n^-$** with one less colour



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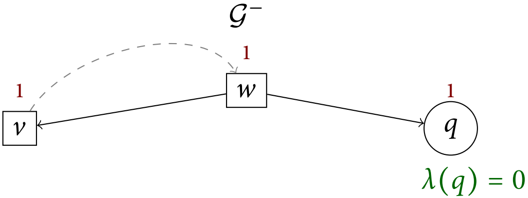

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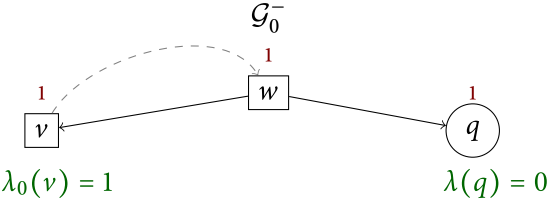

Unfolding Parity Games

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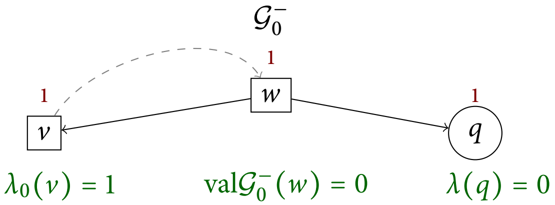

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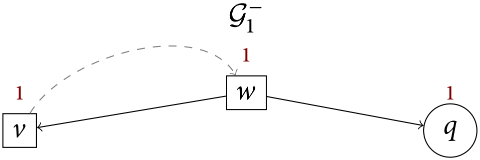

$\lambda_0^-(v) = 1$ $\quad$ $\mathrm{val}\mathcal{G}_0^-(w) = 0$ $\quad$ $\lambda(q) = 0$

$\lambda_1^-(v) = 0$

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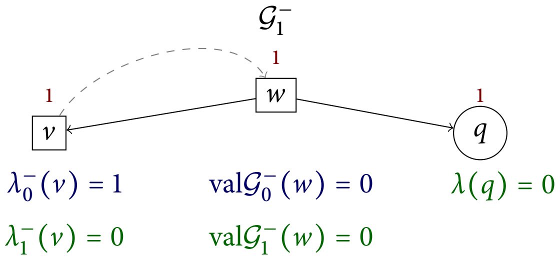
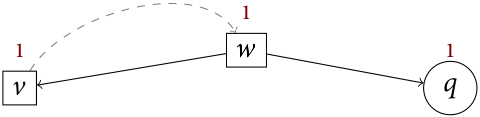


Unfolding of a game $\mathcal{G}$: **sequence of games** $\mathcal{G}_n^-$ with one less colour

$\mathcal{G}_2^-$



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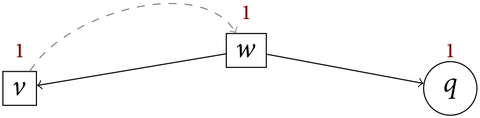
Unfolding Parity Games

Unfolding of a game $\mathcal{G}$: **sequence of games $\mathcal{G}_n^-$** with one less colour

$\mathcal{G}_*^- = \mathcal{G}_2^- = \mathcal{G}_1^-$



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$$\lambda_*(v) = \lambda_2(v) = 0 = \mathrm{val}\mathcal{G}_*^-(v) = \mathrm{val}\mathcal{G}_2^-(v)$$

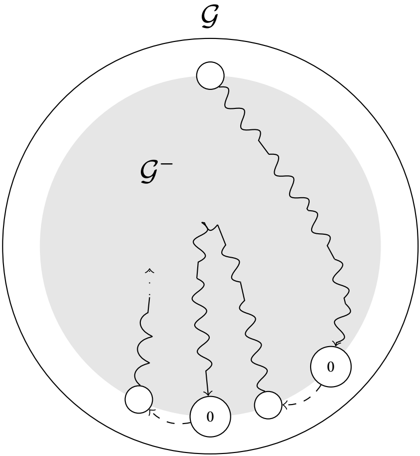
Strategy Construction for Player 0

Goal: $\sup_\sigma \inf_\rho p_{\mathcal{G}} \geq \sup_\sigma \inf_\rho p_{\mathcal{G}^-}$

- take σ optimal in $\mathcal{G}^-$
- construct $\overline{\sigma}$ in $\mathcal{G}$: forget history after position with colour 0
- by contradiction: get ρ in $\mathcal{G}$:

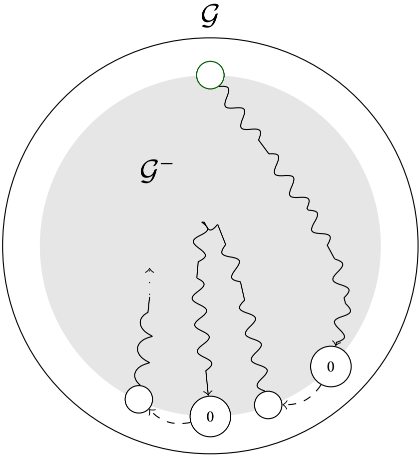
$$p_{\mathcal{G}}(\pi(\overline{\sigma},\rho)) < p_{\mathcal{G}^-}(\pi(\sigma,\rho))$$

- play π and $\overline{\pi}$ together:



σ winning in $\mathcal{G}^-$ $\rightsquigarrow$ $\overline{\sigma}$ winning in $\mathcal{G}$

- take σ optimal in $\mathcal{G}^-$
- construct $\overline{\sigma}$ in $\mathcal{G}$: forget history after position with colour 0
- by contradiction: get ρ in $\mathcal{G}$:
  - $\pi(\sigma, \rho)$ winning in $\mathcal{G}^-$
  - $\pi(\overline{\sigma}, \rho)$ losing in $\mathcal{G}$
- play π and $\overline{\pi}$ together:



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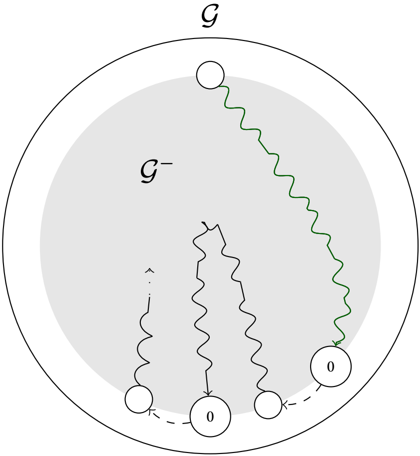

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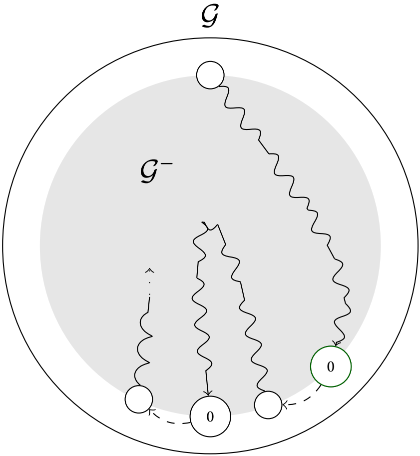
- play π and $\overline{\pi}$ together:
- inevitably reach colour 0



σ **winning in $\mathcal{G}^-$ $\rightsquigarrow$ $\overline{\sigma}$ winning in $\mathcal{G}$**

- take σ optimal in $\mathcal{G}^-$
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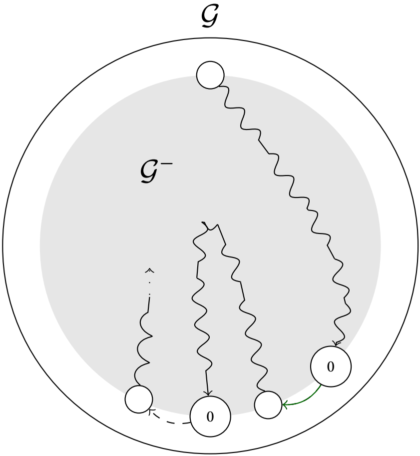
- by contradiction: get ρ in $\mathcal{G}$:
  - $\pi(\sigma, \rho)$ winning in $\mathcal{G}^-$
  - $\pi(\overline{\sigma}, \rho)$ losing in $\mathcal{G}$
- play π and $\overline{\pi}$ together:
- inevitably reach colour 0
- move where σ winning again



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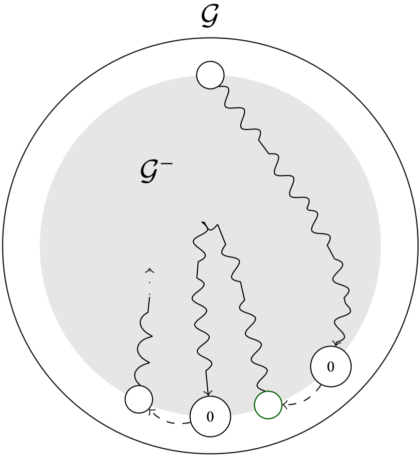

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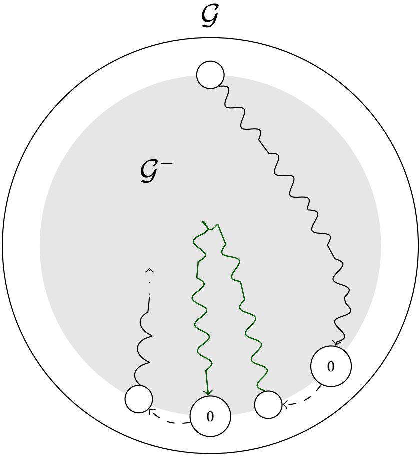

Strategy Construction for Player 0

σ winning in $\mathcal{G}^-$ $\rightsquigarrow$ $\overline{\sigma}$ winning in $\mathcal{G}$

- take σ optimal in $\mathcal{G}^-$
- construct $\overline{\sigma}$ in $\mathcal{G}$: forget history after position with colour 0
- by contradiction: get ρ in $\mathcal{G}$:
  - $\pi(\sigma, \rho)$ winning in $\mathcal{G}^-$
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- play π and $\overline{\pi}$ together:
- inevitably reach colour 0
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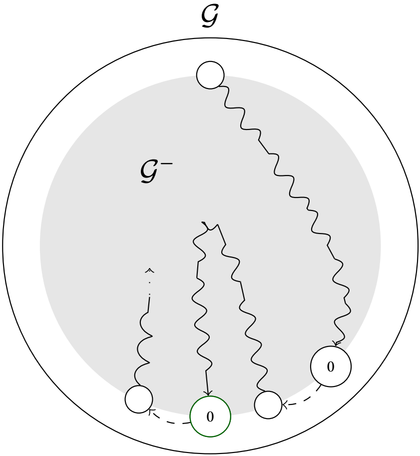

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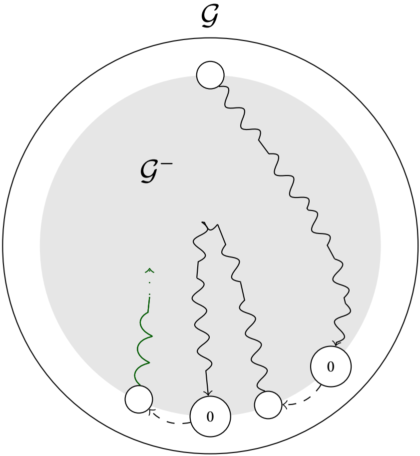
- inevitably again reach 0
- …continue and win in $\mathcal{G}$ ($\perp$)



Strategy Construction for Player 1

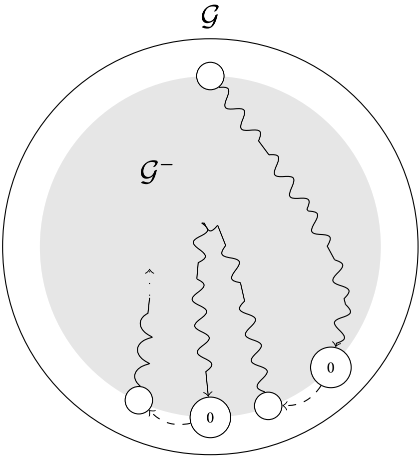
Goal: $\inf_\rho \sup_\sigma p_{\mathcal{G}} \leq \inf_\rho \sup_\sigma p_{\mathcal{G}^-}$

Construct $\overline{\rho}$ in $\mathcal{G}$ from v by taking ρ_i from the **first** $\mathcal{G}_i^-$ where Player 1 wins from v.

- by contradiction: get ρ in $\mathcal{G}$:

  $$p_{\mathcal{G}}(\pi(\sigma, \overline{\rho})) > p_{\mathcal{G}^-}(\pi(\sigma, \rho_{i_0}))$$

- play π and $\overline{\pi}$:



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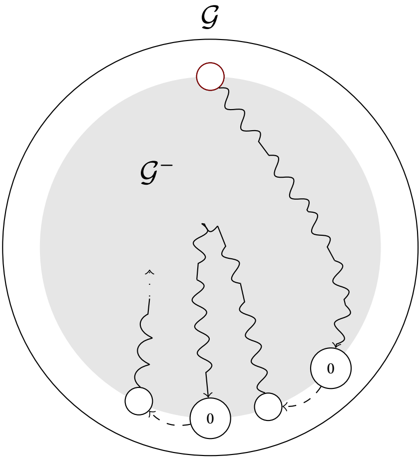

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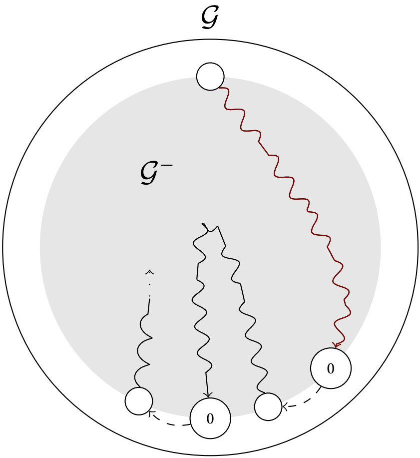

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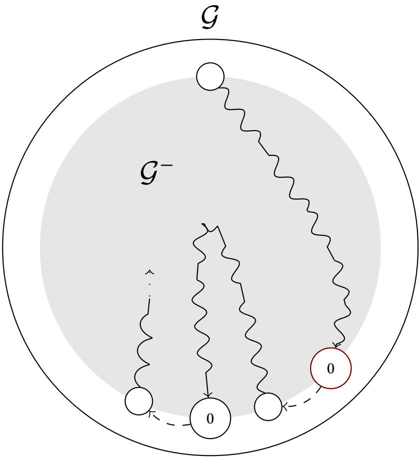

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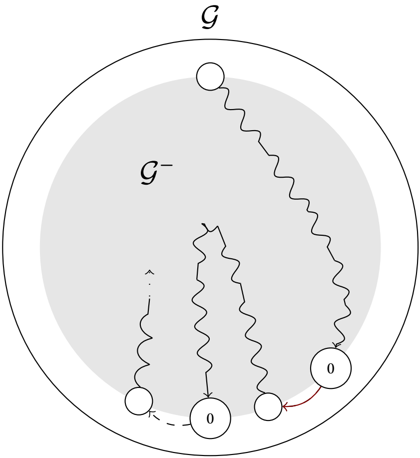

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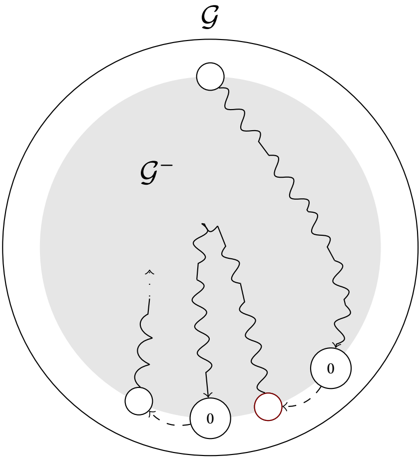

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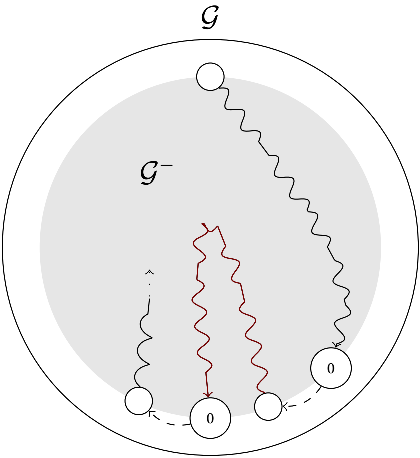

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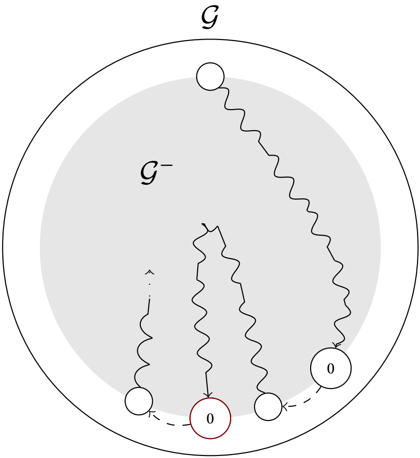

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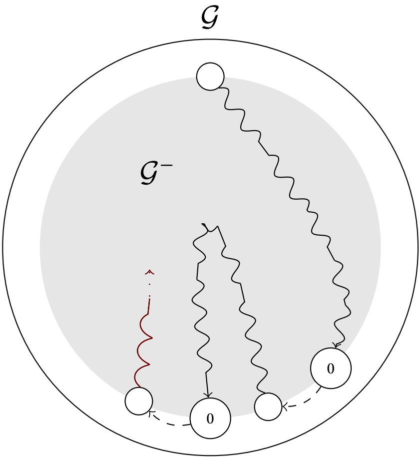

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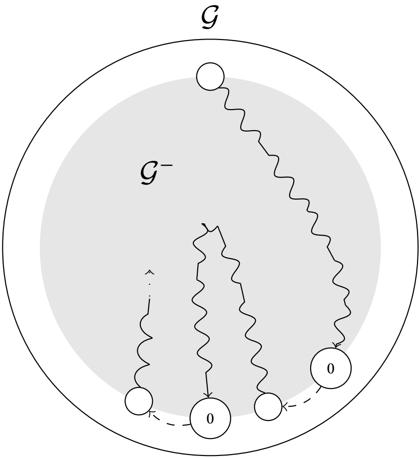
- impossible! ($\lambda_0(v) = 1$)



Consequences

Positional Determinacy of Parity Games

- observe that **reachability games are positionally determined**
- forgetting history retains positionality (Player 0)
- combining positional strategies is positional (Player 1)
- constructing strategies for both players **ensures determinacy**

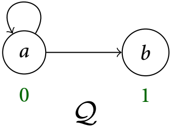
Algorithm for Parity Games

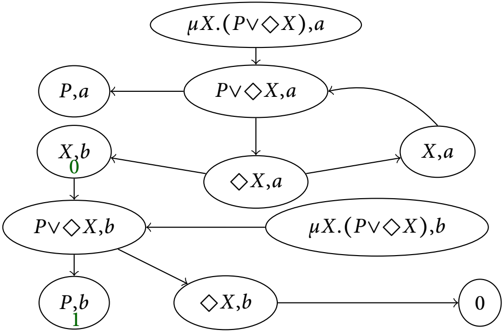
- solve reachability games in linear time by **backwards induction**
- to solve with d colours solve all $\mathcal{G}_j^-$ with $d-1$ colours
- assume there are x_c positions with colour c
- in every unfolding step at least one of x_c positions has λ reset to 0
- calculate time for n positions and d colours:

$$T(d,n) = x_0 \cdot T(d-1, n-x_0) = x_0 \cdot x_1 \cdot T(d-2, n-x_0-x_1) = \dots$$

$$\dots = x_0 \cdot x_1 \cdot \dots \cdot x_{d-1} \le \left(\frac{n}{d}\right)^d$$

Model Checking Games e.g. $\mathrm{MC}[\mathcal{Q}, \varphi]$ for $\mathcal{Q}$ and $\varphi = \mu X.(P \vee \Diamond X)$.





Correctness of Model Checking Games

To evaluate $\mu X.\varphi(X)$ set $X_0 = \varnothing$ and compute $X_{i+1} := \varphi(X_i)$ until $X_{i+1} = X_i$.

- Observe that $\mathrm{MC}[\mathcal{Q}, \mu X.\varphi(X)]_0^-$ coincides with $\mathrm{MC}[\mathcal{Q}, \varphi(X_0)]$.
- **Consequence:** parity games are model checking games for Lμ.

Outlook

Parity Games

- better leave the end nodes
  - or add stopping probabilities?

Unfolding Method:

- natural way to make induction on the number of colours,
- the **first step** to understand any parity-like games,
- lift properties from reachability games almost for free,
- proves connection between μ-calculus and games,
- extends to many other parity-like games:
  - stochastic, quantitative, imperfect information,
  - hints at good algorithmic behaviour.

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Thank You