Parity Games with Partial Information Played on Graphs of Bounded Complexity

MFCS, Brno, 23. Aug. - 27. Aug. 2010

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Algorithmic Synthesis of Reactive Systems

Basic Problem:

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- Given a *plant* $\mathcal{P}$ and a *specification* φ
- Construct a *controller* $\mathcal{C}$ which ensures that all possible system behaviours satisfy φ

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- Construct a *controller* $\mathcal{C}$ which ensures that all possible system behaviours satisfy φ
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  controller (player 0) vs. environment (player 1)
- A *winning strategy* for player 0 yields a controller which ensures a correct system

The Model: Games

Nonterminating systems with finite state space

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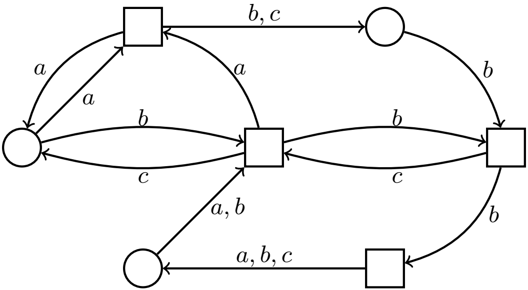
$\rightsquigarrow$

Infinite games on finite graphs

$$G = (V, V_0, \delta : V \times A \to V, \varphi \subseteq (VA)^\omega)$$



The Model: Partial Information

Controller with restricted observational powers:

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$$\mathcal{G} = (G, \sim^V, \sim^A)$$

The Model: Strategies

Strategy for player 0:

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$$f : (\mathsf{VA})^* V_0 \rightarrow A$$

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Strategy for player 0:

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such that

$$f(v_0 a_1 v_1 \ldots a_n v_n) = f(w_0 b_1 w_1 \ldots b_n w_n)$$

if

- $v_i \sim^V w_i$
- $a_i \sim^A b_i$

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$\rightsquigarrow$ Parity games with observable colors

- $\mathrm{col} : V \to \{1, \dots, m\}$
- $\mathrm{col}(u) = \mathrm{col}(v)$ if $u \sim^V v$
- player 0 wins, if the least color seen infinitely often is even

Complexity

Complexity

Theorem

Solving parity games with partial information and two observable colors is EXPTIME*-complete. (Reif '84)*

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Solving parity games with full information and m colors is in PTIME.

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Approach:
Optimize Powerset Construction (Reif '84)

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- On-the-fly construction

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- Symbolic/compressed data representation
- On-the-fly construction

- May be faster in praxis
- No better complexity class

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- What is the impact of graph complexity?

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- What is the impact of graph complexity?

Solving full information parity games is not known to be in PTIME, but on graphs of bounded complexity, it is!

Graph-Complexity

Applicable to parity games with *full information* played on graphs of bounded

- tree-width (Obdrzalek, '03)
- DAG-width (Berwanger et al, '06)
- Kelly-width (Hunter, Kreutzer '08)
- ...

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What about parity games with partial information?

- Partial information inhibits a direct dynamic approach
- Powerset Construction yields an exponentially larger graph in general
- But: if the original graph is simple, we might hope that the new graph is small/simple

Graph-Complexity

We consider:

- tree-width
- entanglement
- directed path-width
- DAG-width

Graph-Complexity

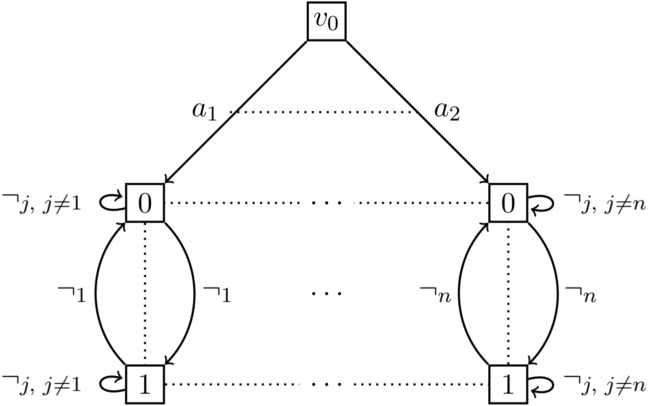
- All measures we consider can be characterized in terms of cops and robber games
- Several cops try to catch one robber on the graph
- The cop player places cops on graph vertices
- Before the cops actually land, the robber may run along cop-free edges of the graph
- number of cops needed to catch the robber determines the complexity

Exponential Growth of Measures

Theorem

There is a class $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of games with partial information and with complexity ≤ 2 such that the powerset graphs $\overline{G}_n$ have exponential complexity in the size of $\mathcal{G}_n$.

Exponential Growth of Measures



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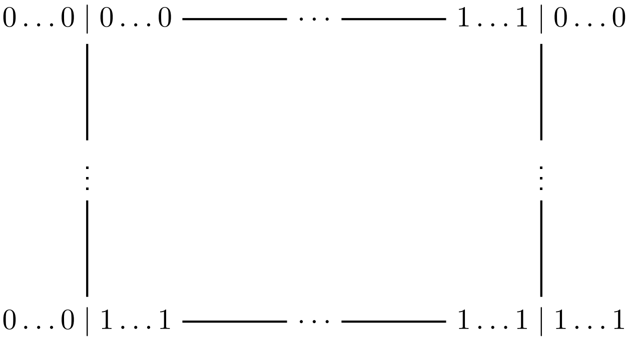

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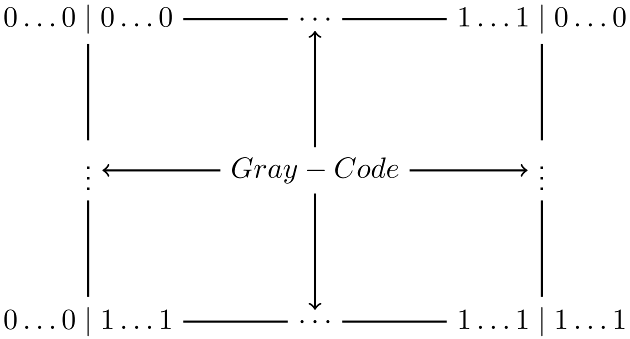

Intrinsic Complexity

- The powerset construction does neither yield small nor simple graphs when performed on graphs of bounded complexity

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- The powerset construction does neither yield small nor simple graphs when performed on graphs of bounded complexity
- Can we find other approaches to exploit simplicity of graphs?

Intrinsic Complexity

Theorem

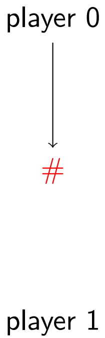
Solving parity games with partial information and two observable colors is EXPTIME*-hard on graphs of entanglement and directed path-width at most* 2.

Idea

Based on the original idea for $\textsc{Exptime}$-hardness of the general case (Reif '84)

- Any $L \in \textsc{Exptime}$ is recognized by a TM M of the following kind
  - alternating
  - polynomial space bound
  - one tape
- Given an input u, player 0 guesses an accepting computation of M on u *character by character*
- Player 1 checks that the construction is correct:
  any configuration only differs *locally* from the previous one

Idea



Idea

player 0

↓

$\#(q_0 u_0)$

player 1

Idea

player 0

$\#(q_0u_0)u_1$

player 1

Idea

player 0

$\# (q_0 u_0) u_1 u_2$

player 1

Idea

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$\#(q_0 u_0) u_1 u_2 u_3$

player 1

Idea

player 0

$$\#(q_0u_0)u_1u_2u_3 \cdots \sqcup\sqcup\#$$

player 1

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$$\#(q_0u_0)u_1u_2u_3 \;\cdots\; \sqcup\sqcup\# \cdots \#$$

player 1

Idea

player 0

$$\#(q_0u_0)u_1u_2u_3 \;\cdots\; \sqcup\sqcup\# \cdots \# \cdots \sigma_{i-2}\,\sigma_{i-1}\sigma_i$$

player 1

Idea

player 0

$\#(q_0 u_0) u_1 u_2 u_3 \;\cdots\; \llcorner\!\lrcorner\,\llcorner\!\lrcorner\,\# \cdots \# \cdots\; \sigma_{i-2}\, \sigma_{i-1} \sigma_i\, \sigma_{i+1}$

player 1

Idea

player 0

$$\#(q_0u_0)u_1u_2u_3 \;\cdots\; \sqcup\sqcup\# \cdots \# \cdots \;\sigma_{i-2}\,\sigma_{i-1}\sigma_i\,\sigma_{i+1}$$

player 1

Idea

player 0

↓

$\# (q_0 u_0) u_1 u_2 u_3 \ \cdots \ \llcorner\!\lrcorner \llcorner\!\lrcorner \# \cdots \# \cdots \ \sigma_{i-2} \ \sigma_{i-1} \sigma_i \ \sigma_{i+1} \ \sigma_{i+2}$

player 1

Idea

player 0

$$\cdots \ \# \ \cdots \sigma_{i-1} \, \sigma_i \, \sigma_{i+1} \cdots \ \# \qquad \cdots \sigma'_{i-1}$$

player 1

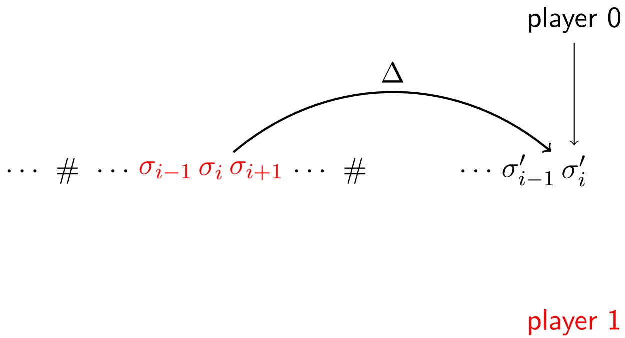
Idea

player 0

$\cdots \; \# \; \cdots \sigma_{i-1}\,\sigma_i\,\sigma_{i+1} \cdots \; \# \qquad \cdots \sigma'_{i-1}\,\sigma'_i$

player 1

Idea


player 0
Δ
$\cdots \# \cdots \sigma_{i-1}\, \sigma_i\, \sigma_{i+1} \cdots \#$
$\cdots \sigma'_{i-1}\, \sigma'_i$
player 1

Idea

- Partial information?

Idea

- Partial information?
- Player 0 must not notice when player 1 turns red!

Playing it on a Simple Graph

(1) Constructing a configuration

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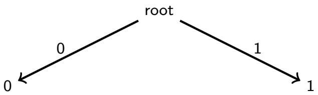
root

Playing it on a Simple Graph

(1) Constructing a configuration



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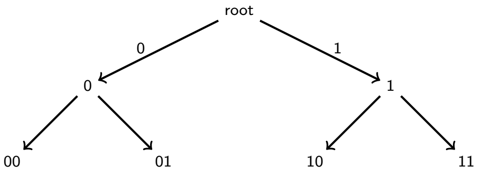

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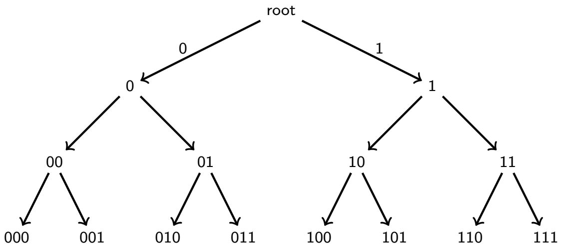

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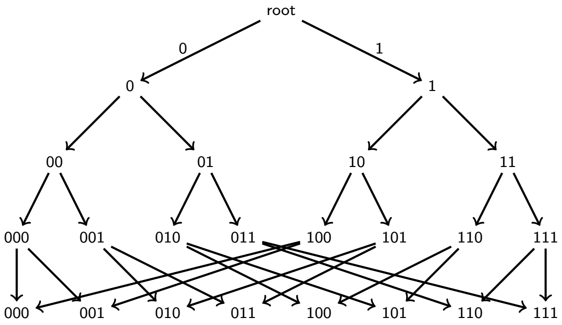

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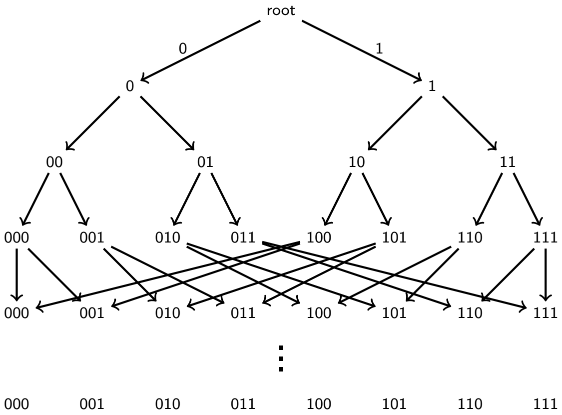

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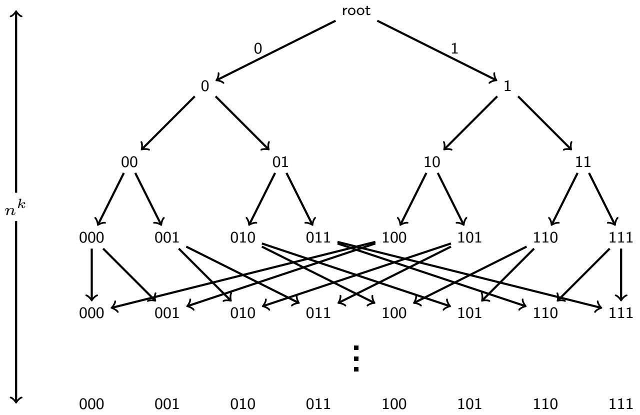

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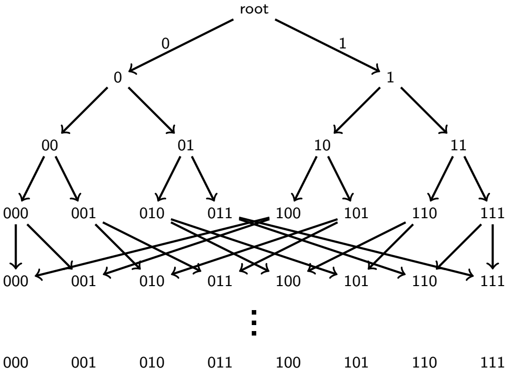
(2) Constructing a sequence of configurations



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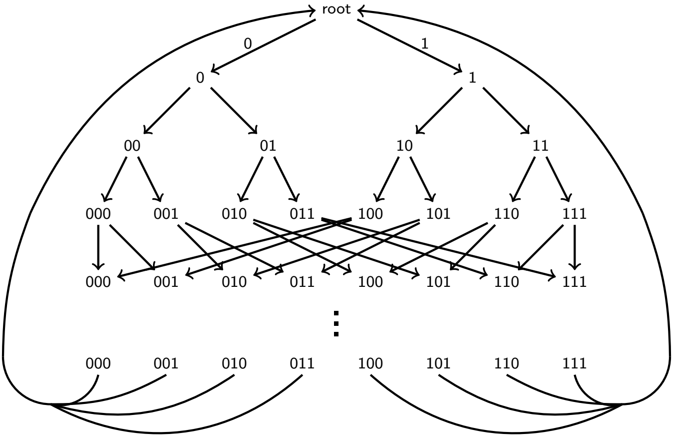

Playing it on a Simple Graph

(1) Constructing a configuration
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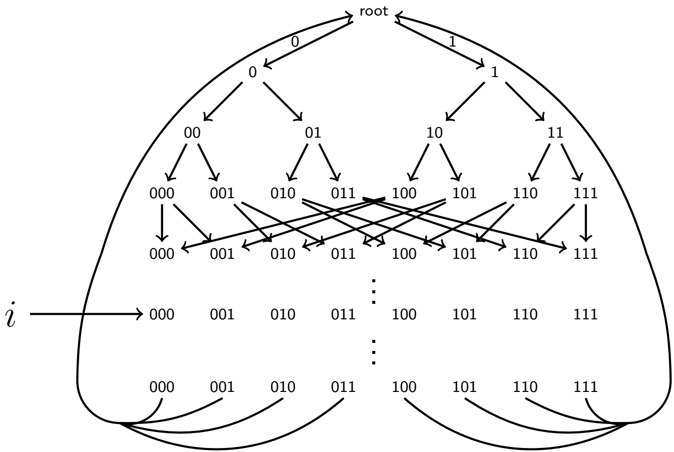

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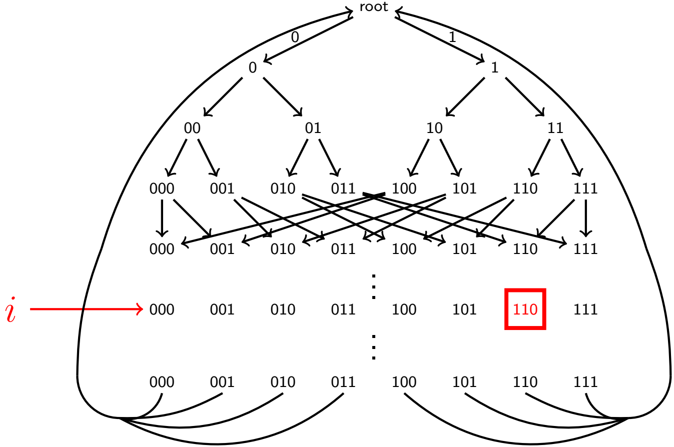

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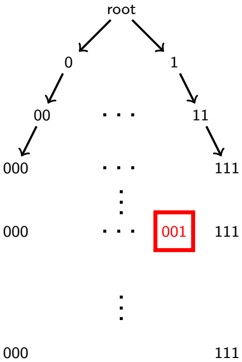

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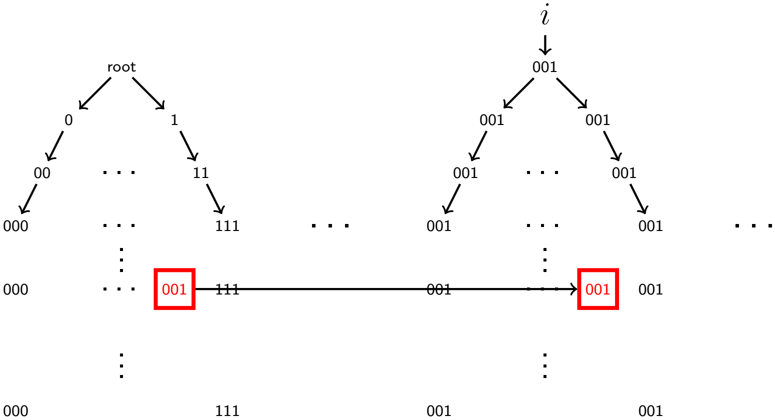

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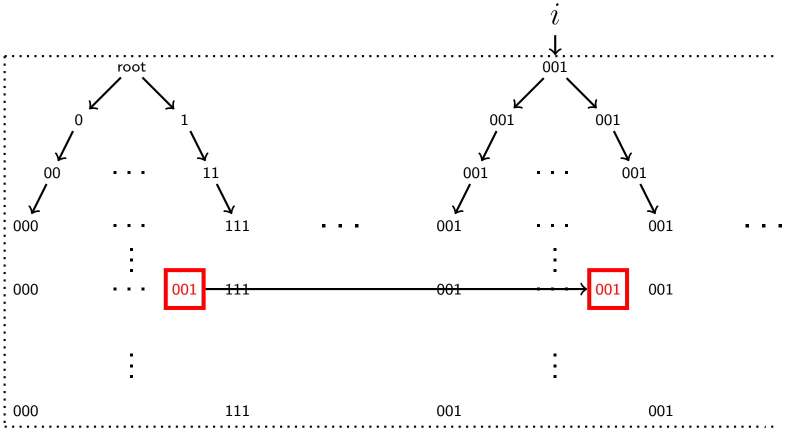

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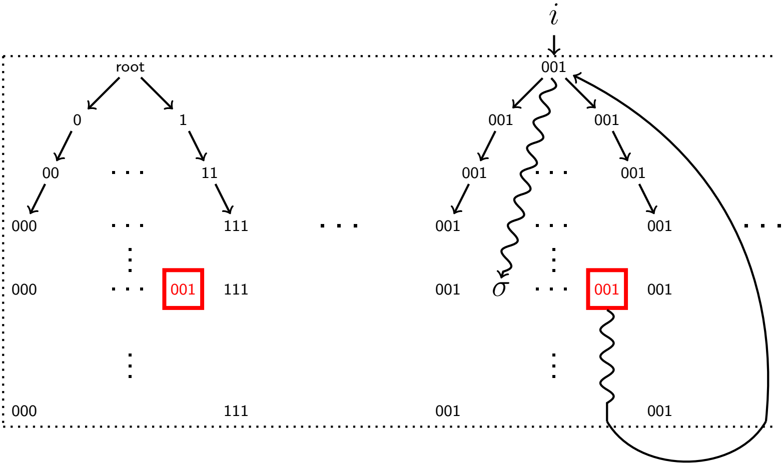

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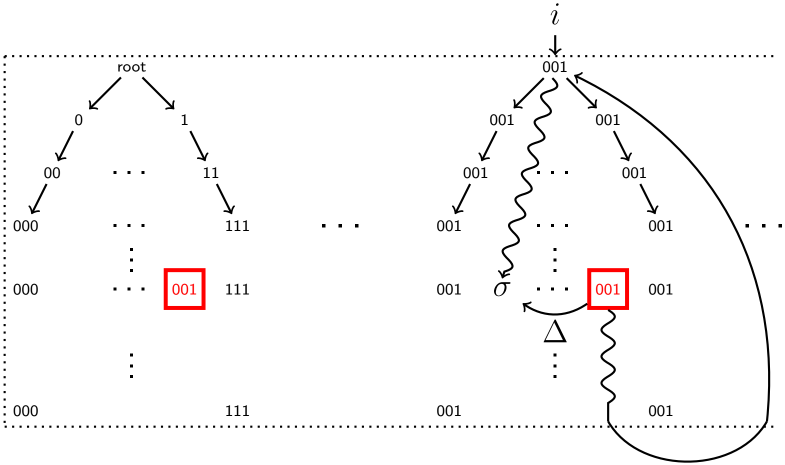

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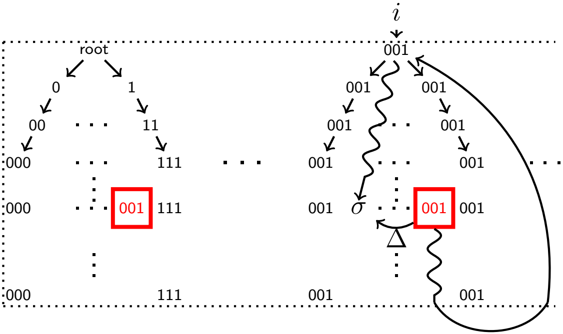

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- Threat of being observed without observing it

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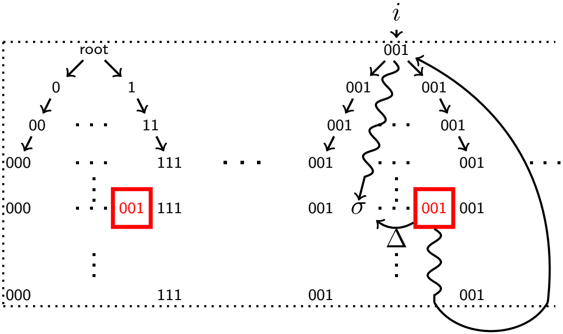

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- Threat of being observed without observing it
- Therefore: no circles between the DAGs
- Directed path-width and entanglement at most 1!

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So far: each configuration has exactly one successor (DTM)

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ATM:

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ATM:

- Alternation is captured by the interaction between player 0 and 1
- Multiple successor configurations increase the graph complexity
- With some technical simplification of the machine: directed path-width and entanglement at most 2!

Intrinsic Complexity

Theorem

Solving parity games with partial information and two observable colors is EXPTIME*-hard on graphs of entanglement and directed path-width at most* 2*.*

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Solving parity games with partial information and two observable colors is PSPACE*-complete on acyclic graphs.*

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- Bounded partial information!
- Each equivalence class of positions has size at most r

Preservation of Measures

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- What about DAG-width?

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Key-Observation:
Paths in the powerset graph $\overline{G}$ can be translated into corresponding paths in the original graph G!

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- $k \cdot r \cdot 2^{r-1}$ cops
- We have to trace several plays on the original graph!

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- The translation does not preserve monotonicity of the cops' strategy!
- Works for directed path-width, as it has monotonicity cost 0
- Works for DAG-width on graphs with strongly connected informations sets

Preservation of Measures

at most r simultaneos plays

$\rightsquigarrow$

at most r robbers that have to be caught simultaneously

Theorem

If r-DAG-width$(G) \leq k$ *then DAG-width*$(\overline{G}) \leq k \cdot 2^{r-1}$.

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at most r simultaneos plays

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If r-DAG-width$(G) \leq k$ *then DAG-width*$(\overline{G}) \leq k \cdot 2^{r-1}$.

Theorem

If tree-width$(G) \leq k$, *then r-DAG-width*$(G) \leq r \cdot k$.

Conjecture

If DAG-width$(G) \leq k$, *then r-DAG-width*$(G) \leq 2 \cdot r \cdot k$.

Complexity

Theorem

Parity games with bounded partial information can be solved in polynomial time on graphs

- *of bounded directed path-width.*
- *of bounded DAG-width, if information sets are strongly connected.*
- *of bounded tree-width.*

Conjecture

Parity games with bounded partial information can be solved in polynomial time on graphs of bounded DAG-width.