Concurrent Graph Searching

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RWTH Aachen University

Computational Complexity

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Here: Solving infinite two-person games on finite graphs.

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- Graph-isomorphism
- Solving parity games

Parity Games

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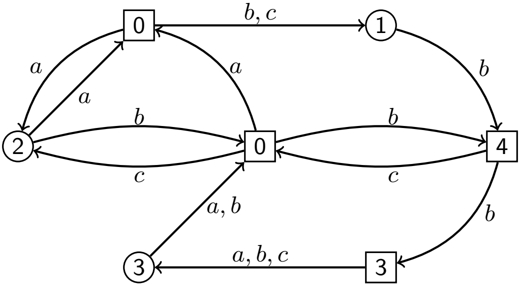

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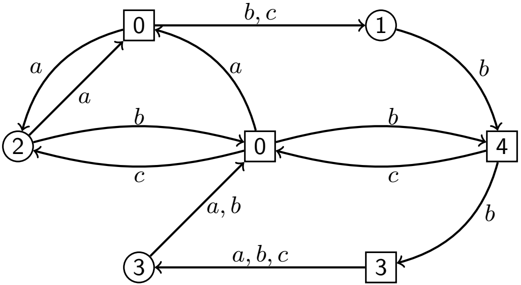

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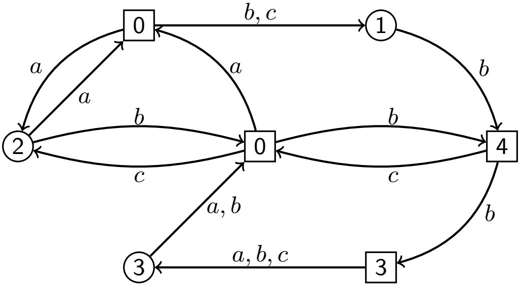

Player 0 wins a play of the game if the least color seen infinitely often is even!

Fundamental Question:

Given a parity game, does player 0 have a winning strategy?

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Parity games can be solved in polynomial time on graphs of bounded tree-width.

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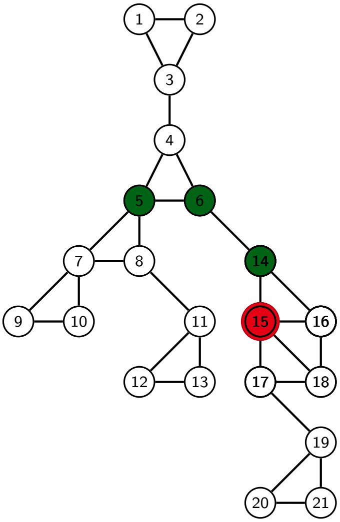
Tree-width can be characterized by cops and robber games...

Graph Complexity



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- several Cops, one Robber
- Cops may be placed arbitrarily on the graph
- Robber reacts by running along cop free paths
- Cops want to capture Robber

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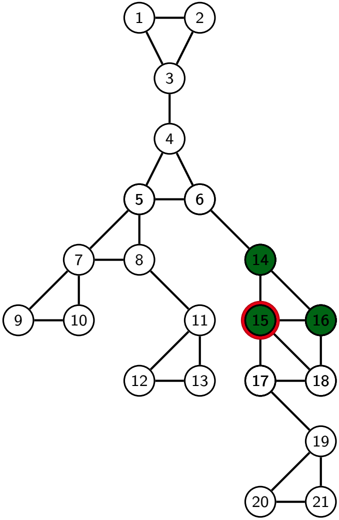

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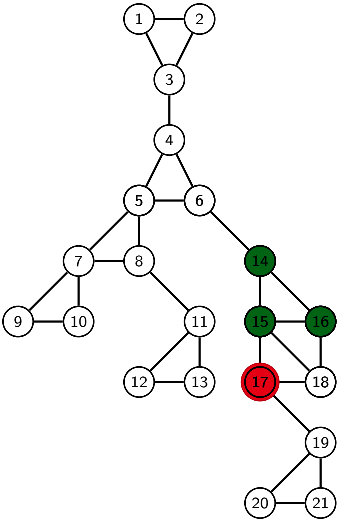

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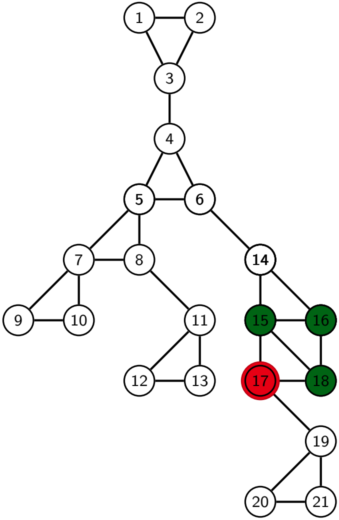

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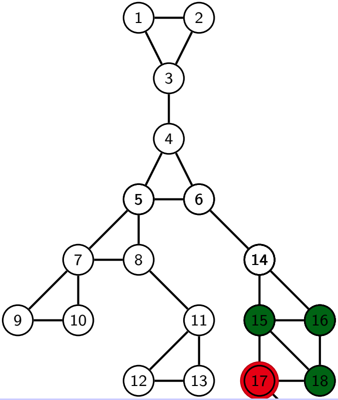

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Theorem (Berwanger et al. 2006)

Parity games can be solved in polynomial time on graphs of bounded DAG-width.

Graph Complexity

How about concurrent problems on graphs?

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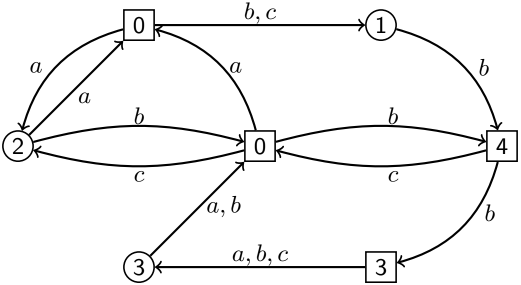

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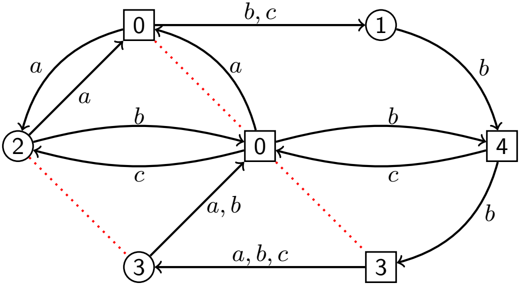

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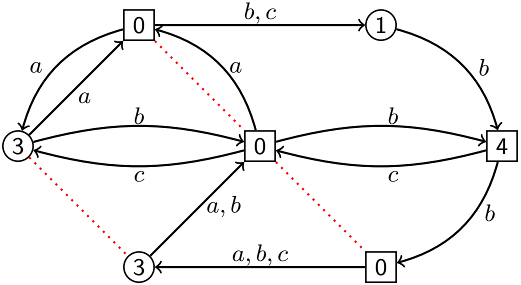

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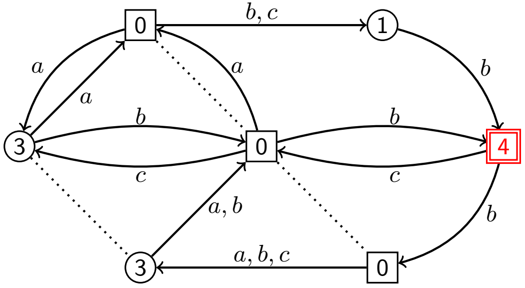

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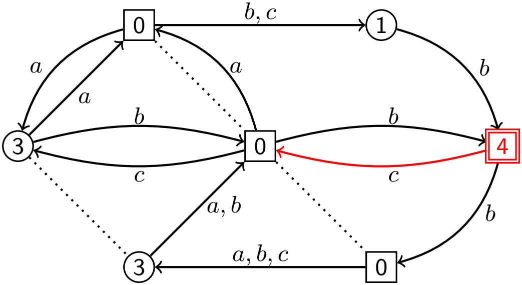

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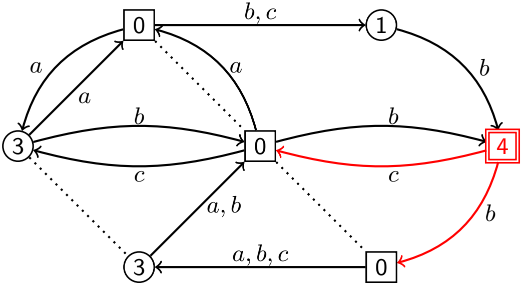

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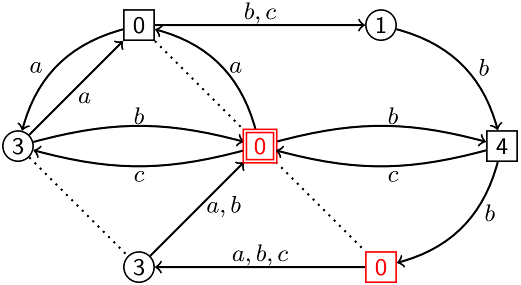

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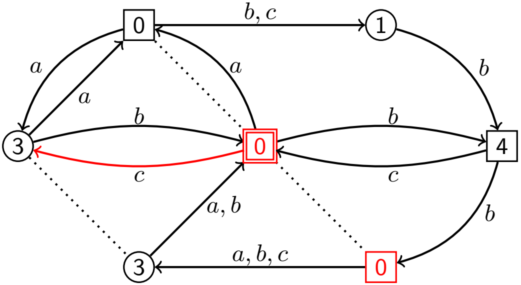

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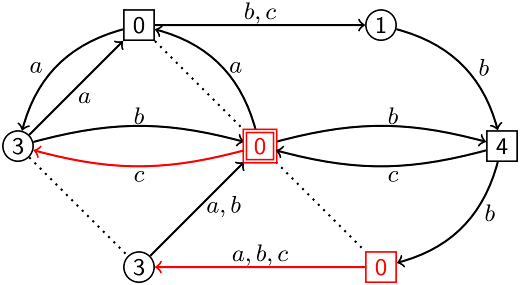

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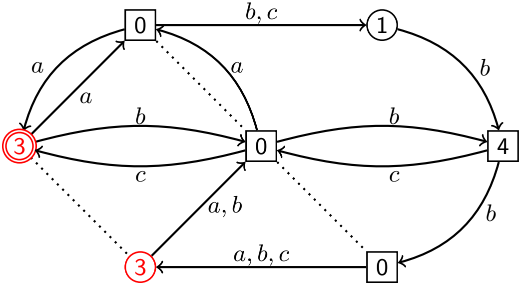

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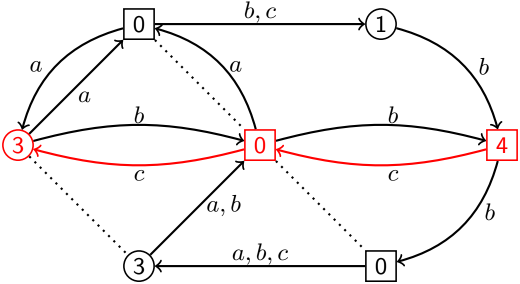

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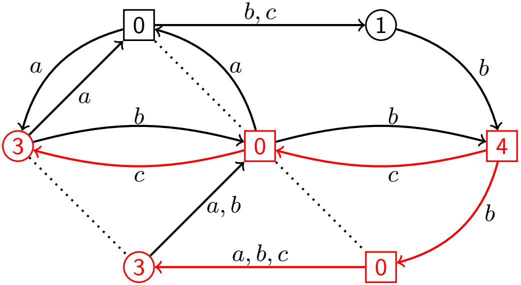

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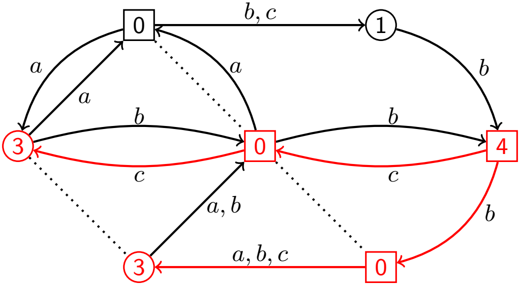

$\rightsquigarrow$ Parity games with imperfect information are inherently concurrent!

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Theorem (P.,Rabinovich 2010)

Parity games with imperfect information are EXPTIME*-hard on graphs of DAG-width at most* 3 *and* PSPACE*-hard on graphs of DAG-width at most* 1.

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- What about bounded imperfect information?
- Each equivalence class of positions has size at most r

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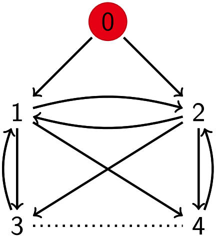

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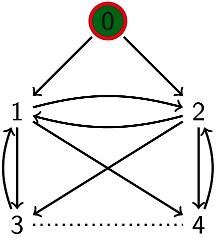

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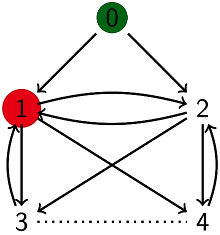

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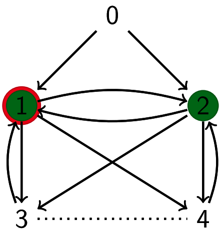

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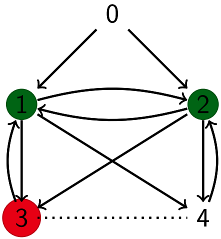

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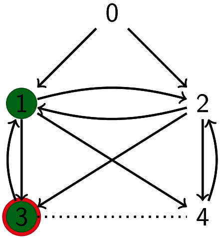

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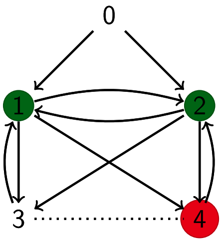

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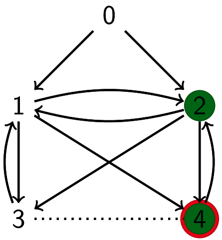

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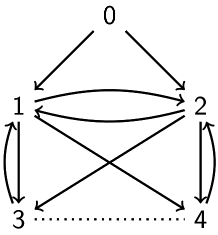

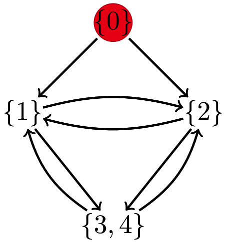

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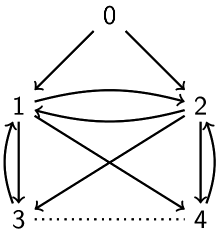

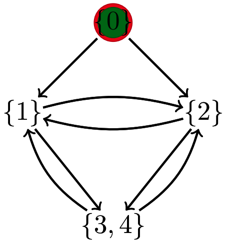

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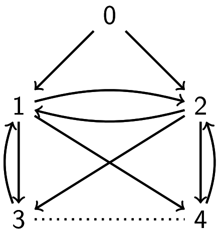

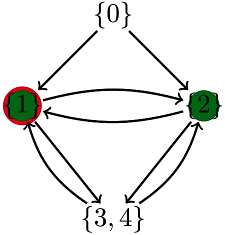

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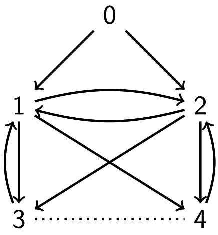

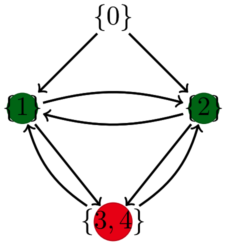

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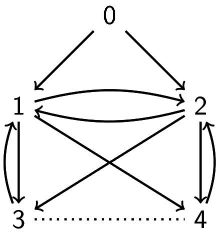

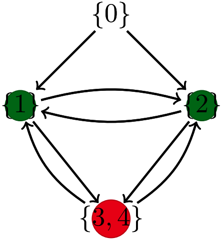

$\rightsquigarrow$ Concurrent graph searching!

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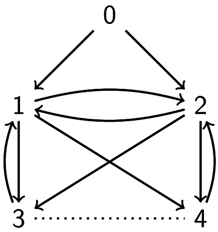

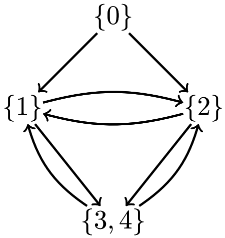

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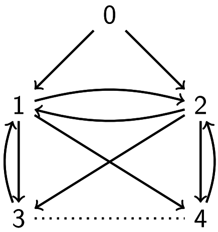

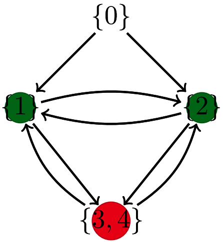

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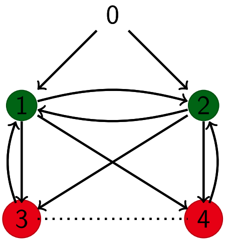

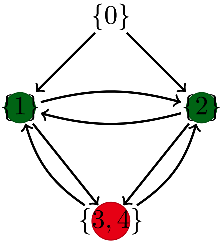

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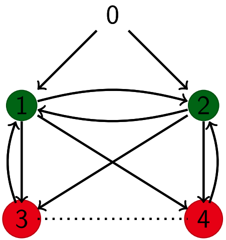

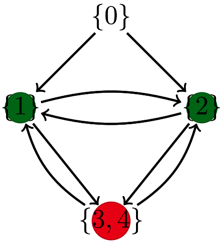

Theorem (P.,Rabinovich 2010)

If $\mathrm{dpw}(G) \leq k$ *then* $\mathrm{dpw}(2^G) \leq k \cdot 2^{r-1}$.

Multiple Robber Games

- Assuming bounded directed path-width is very restrictive
- Having one invisible robber is the same as having unboundedly many visible robbers
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$\rightsquigarrow$ natural hierarchy of complexity values for a graph G:

$$\mathrm{dw}(G) = \mathrm{dw}_1(G) \leq \mathrm{dw}_2(G) \leq \ldots \leq \mathrm{dw}_n(G) = \mathrm{dpw}(G)$$

Multiple Robber Games

Theorem

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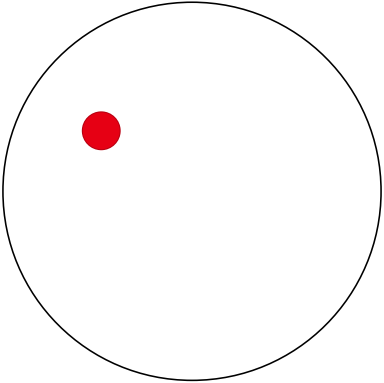

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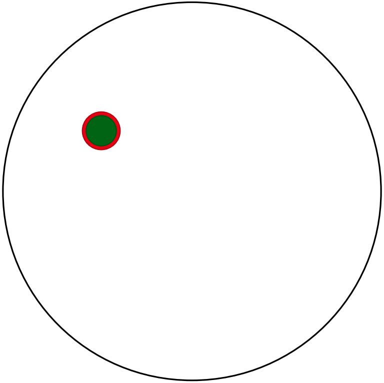

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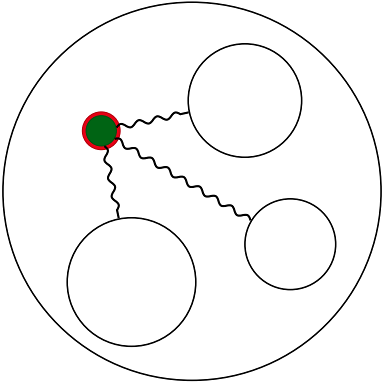

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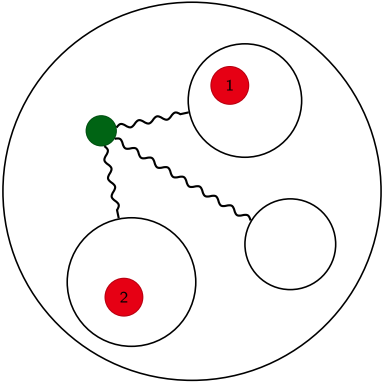

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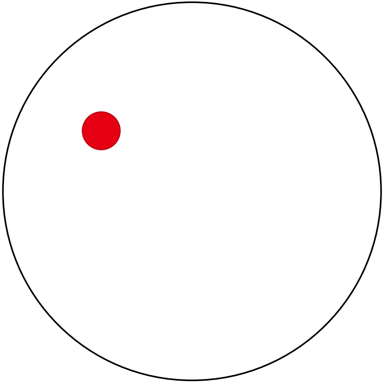

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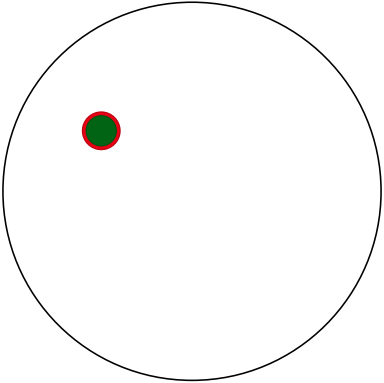

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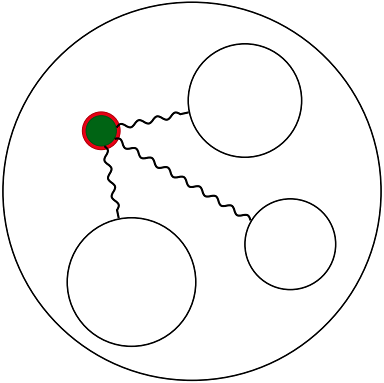

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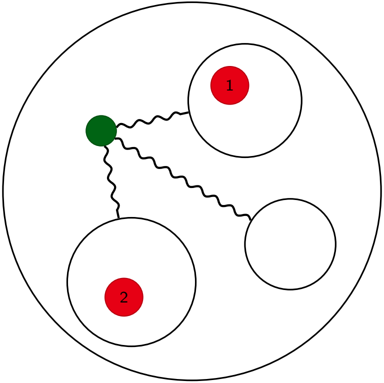

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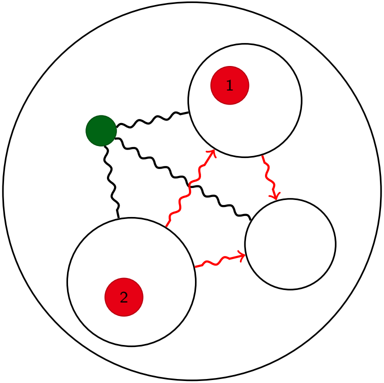

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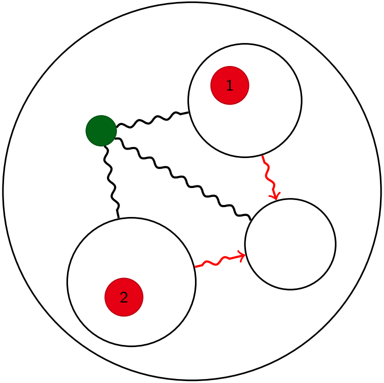

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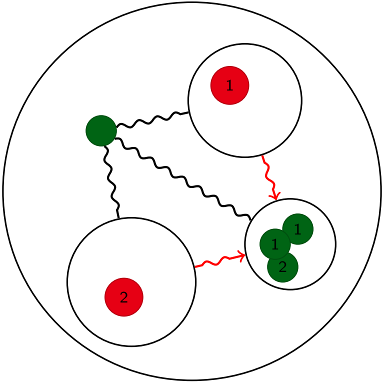

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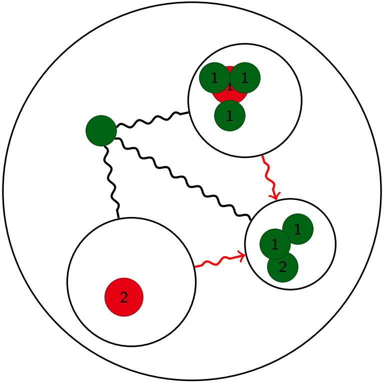

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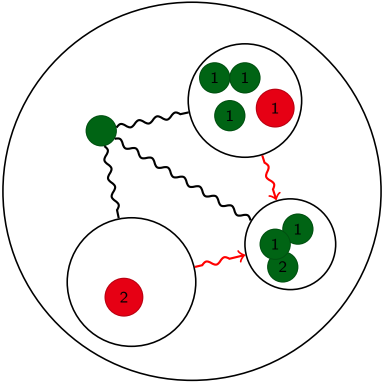

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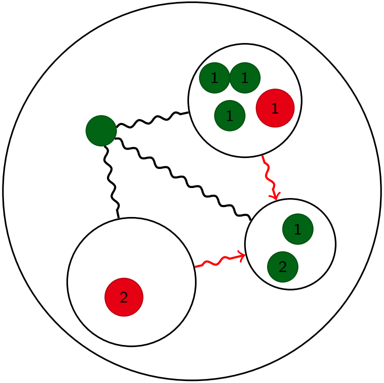

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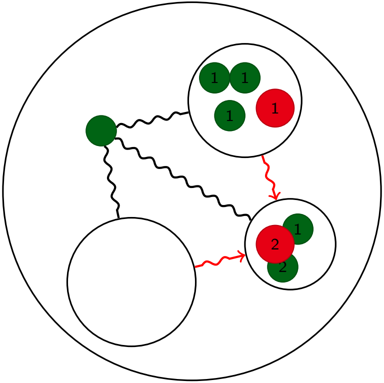


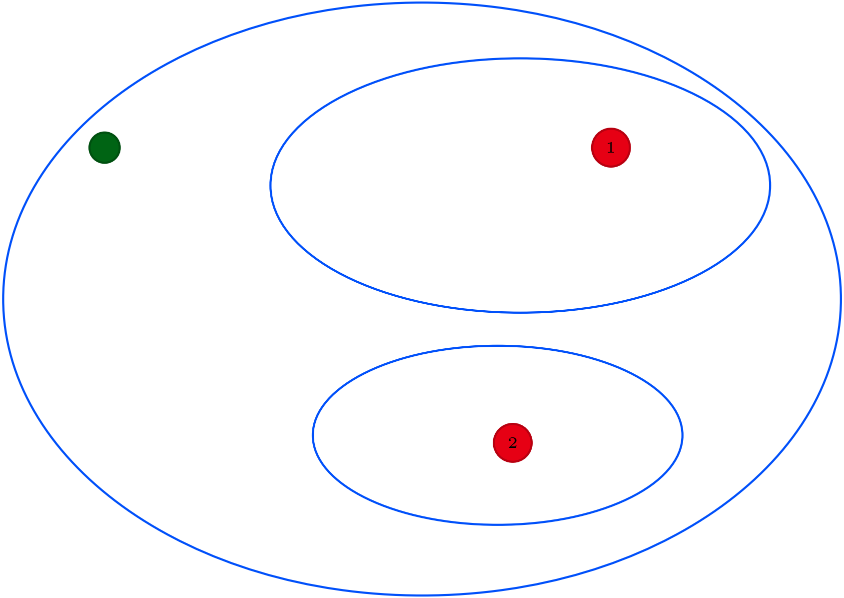
Componentwise Hunting

Theorem

There are graphs G_n, $n \in \mathbb{N}$, such that:

- $\mathrm{dw}(G_n) \leq 4$ *for all* $n \in \mathbb{N}$
- *Componentwise hunting a robber requires at least* n *cops.*

Hunting Multiple Robbers



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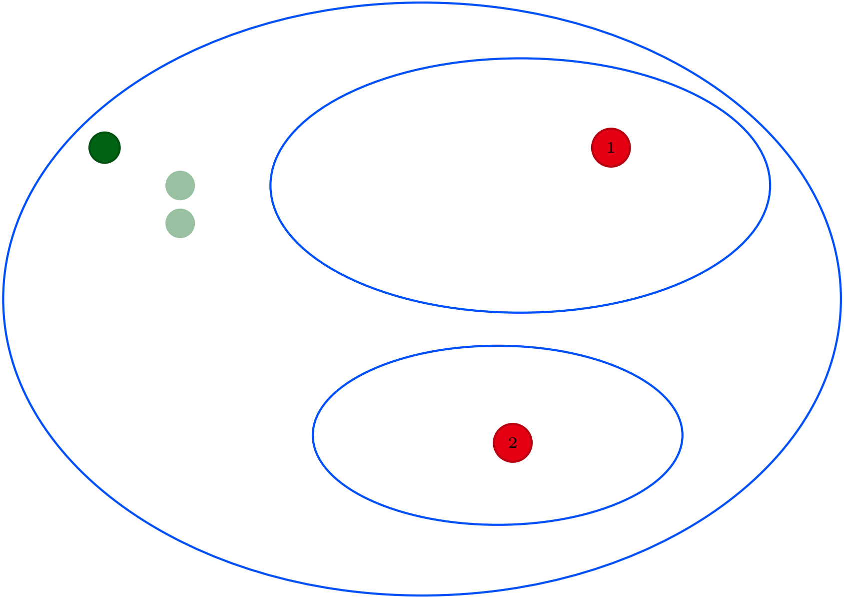

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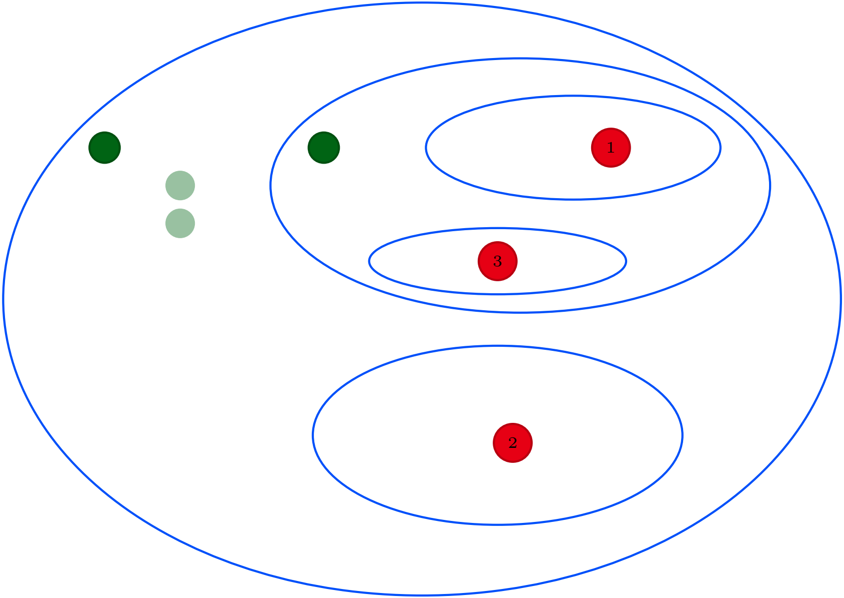

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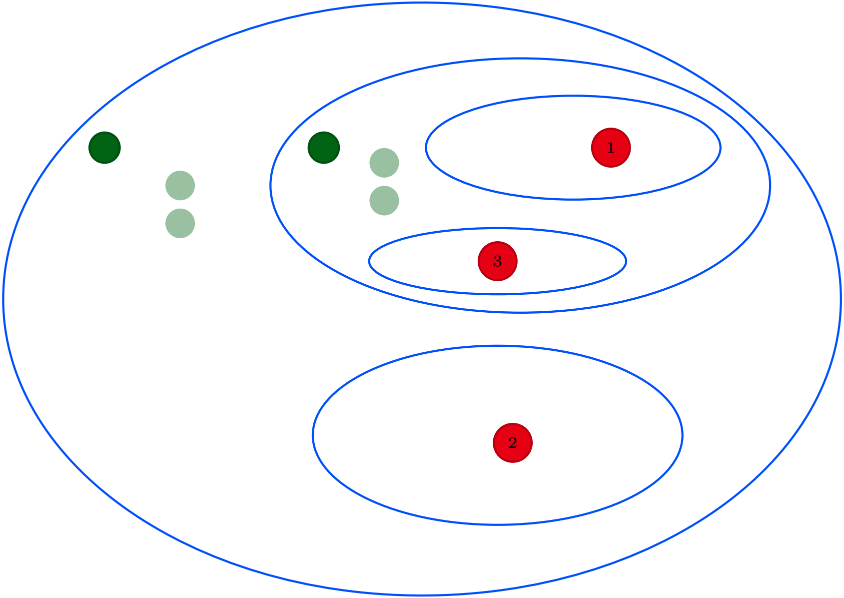

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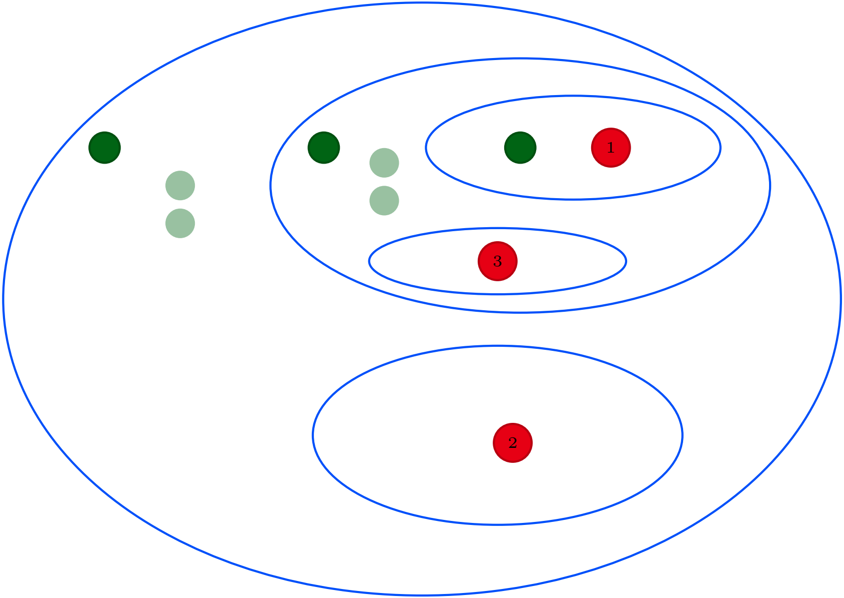

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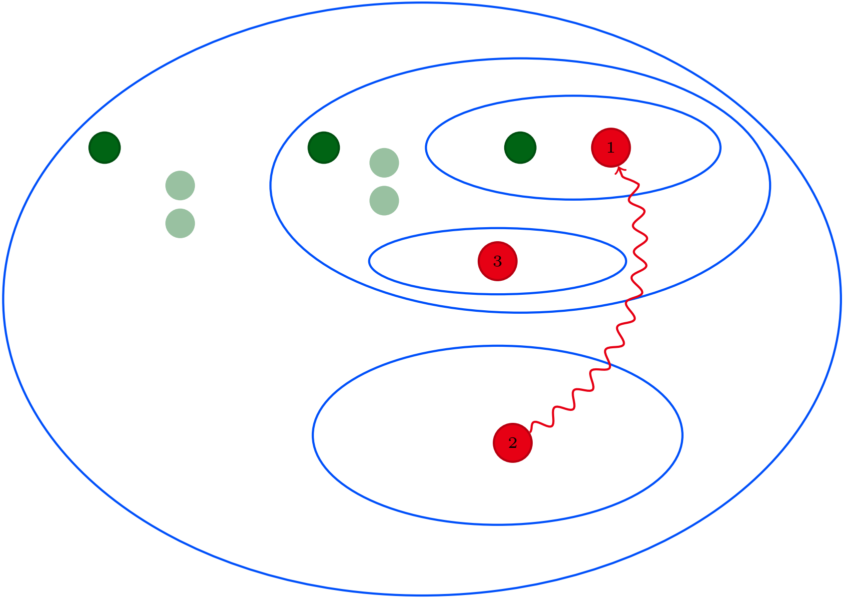

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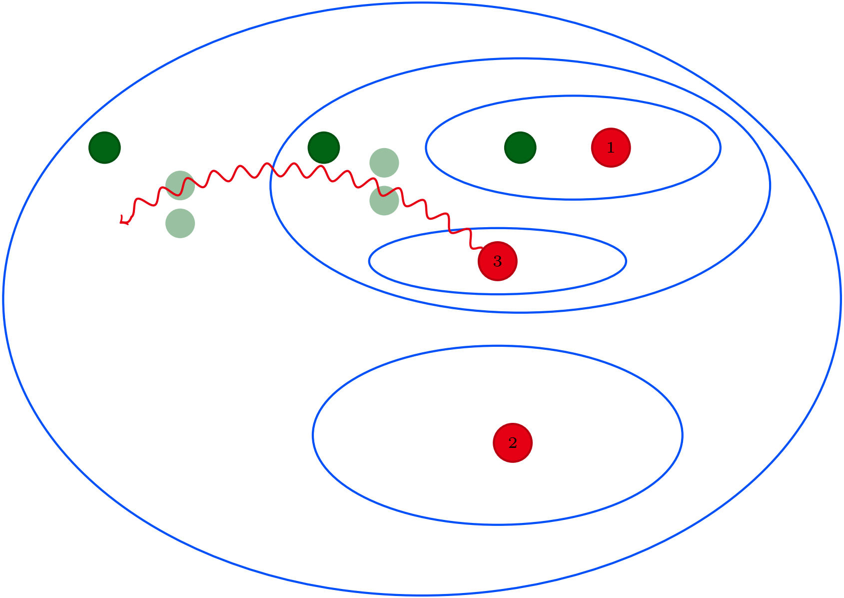

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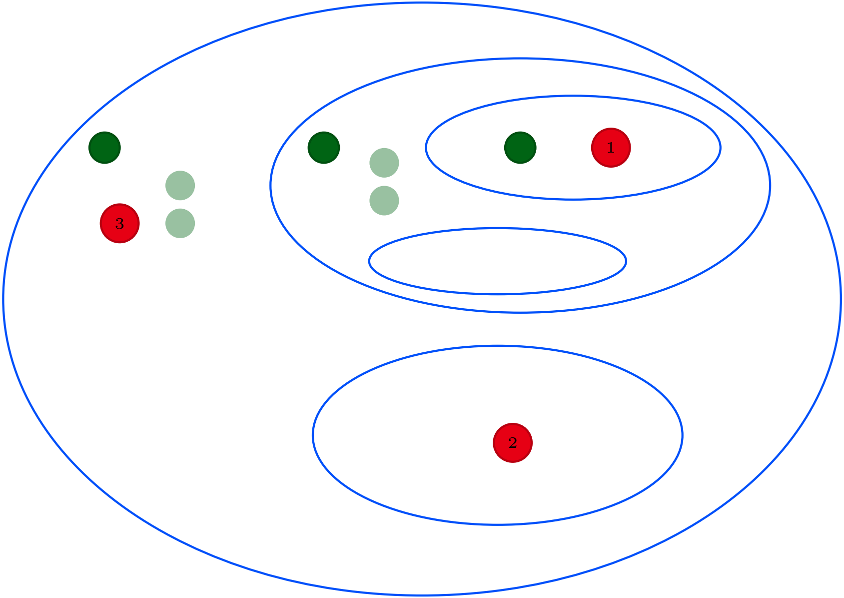

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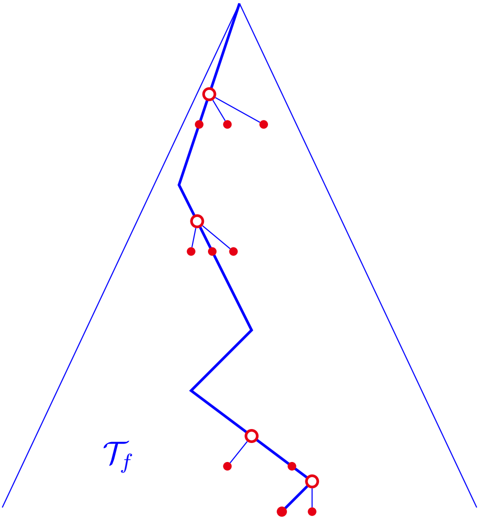

Application

Corollary

Parity games with bounded imperfect information can be solved in polynomial time on graphs of bounded DAG-width.