# Parity Games with Imperfect Information on Graphs of Bounded Complexity

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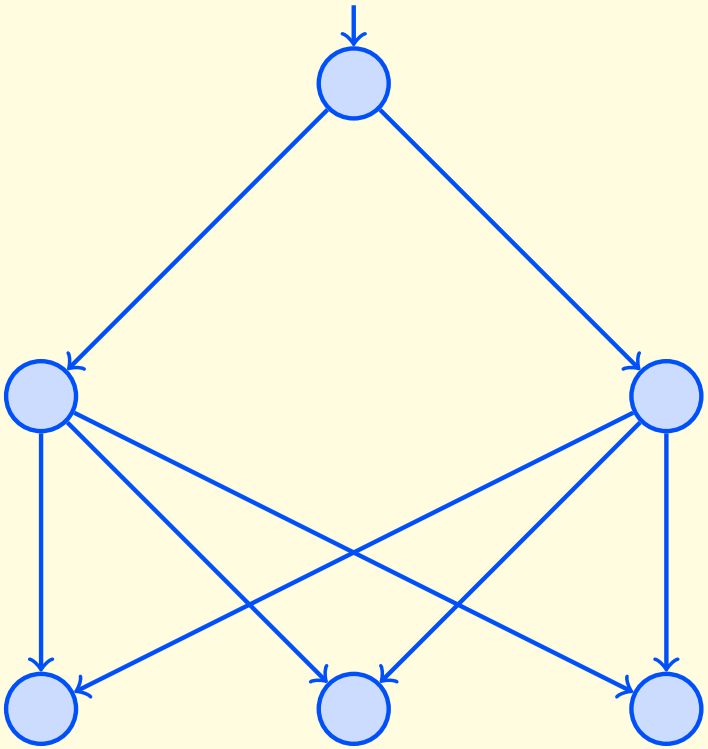
# Imperfect Information



- Parity games:
  $(V, V_0, v, (E_a)_{a \in A}, \Omega)$
- Indistinguishable vertices build an information set.

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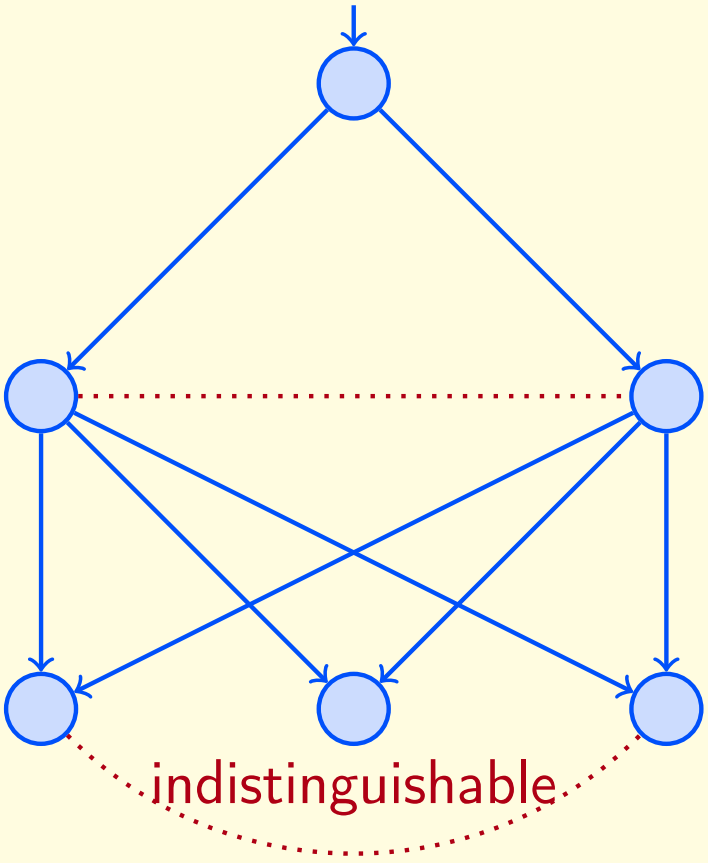


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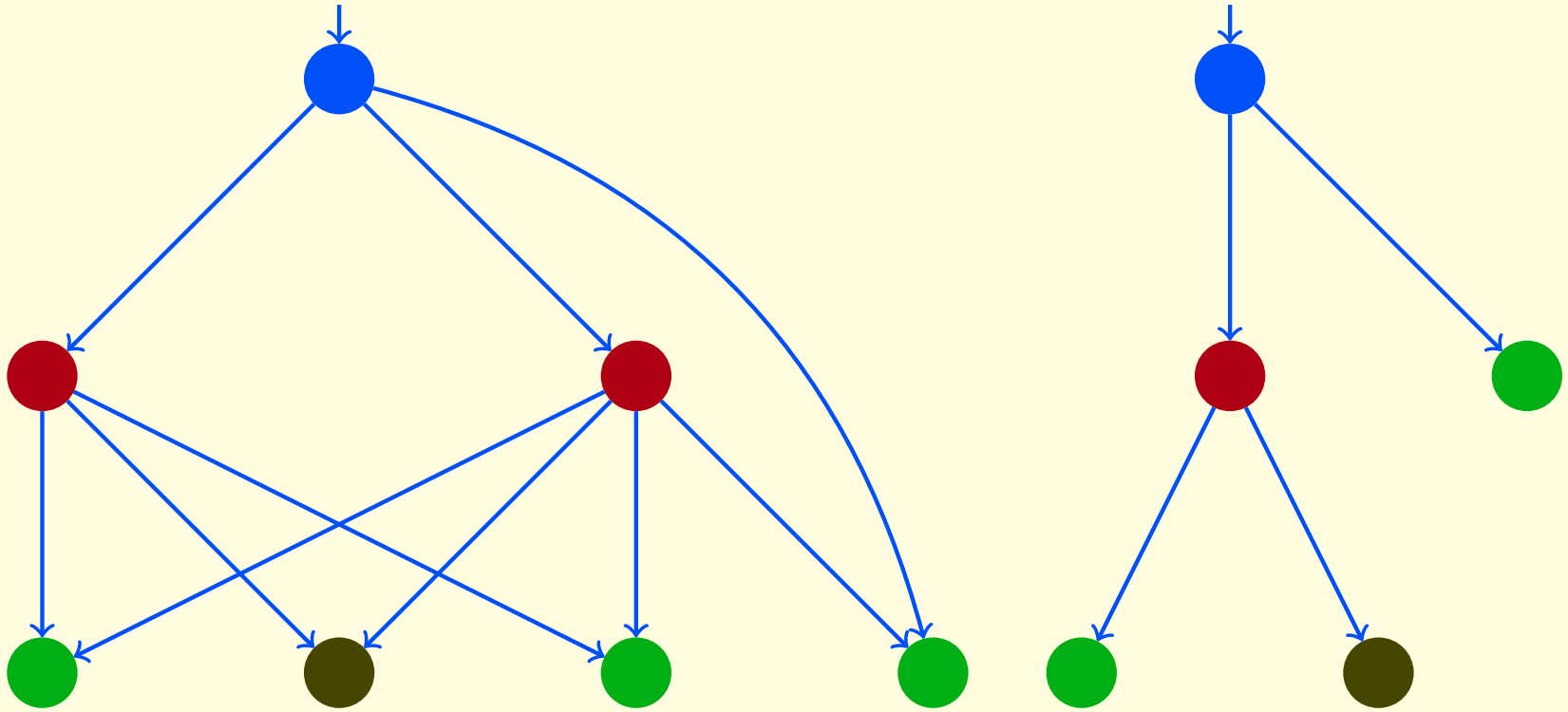
# Powerset Construction (Reif 1984)

Information tracking with perfect recall:



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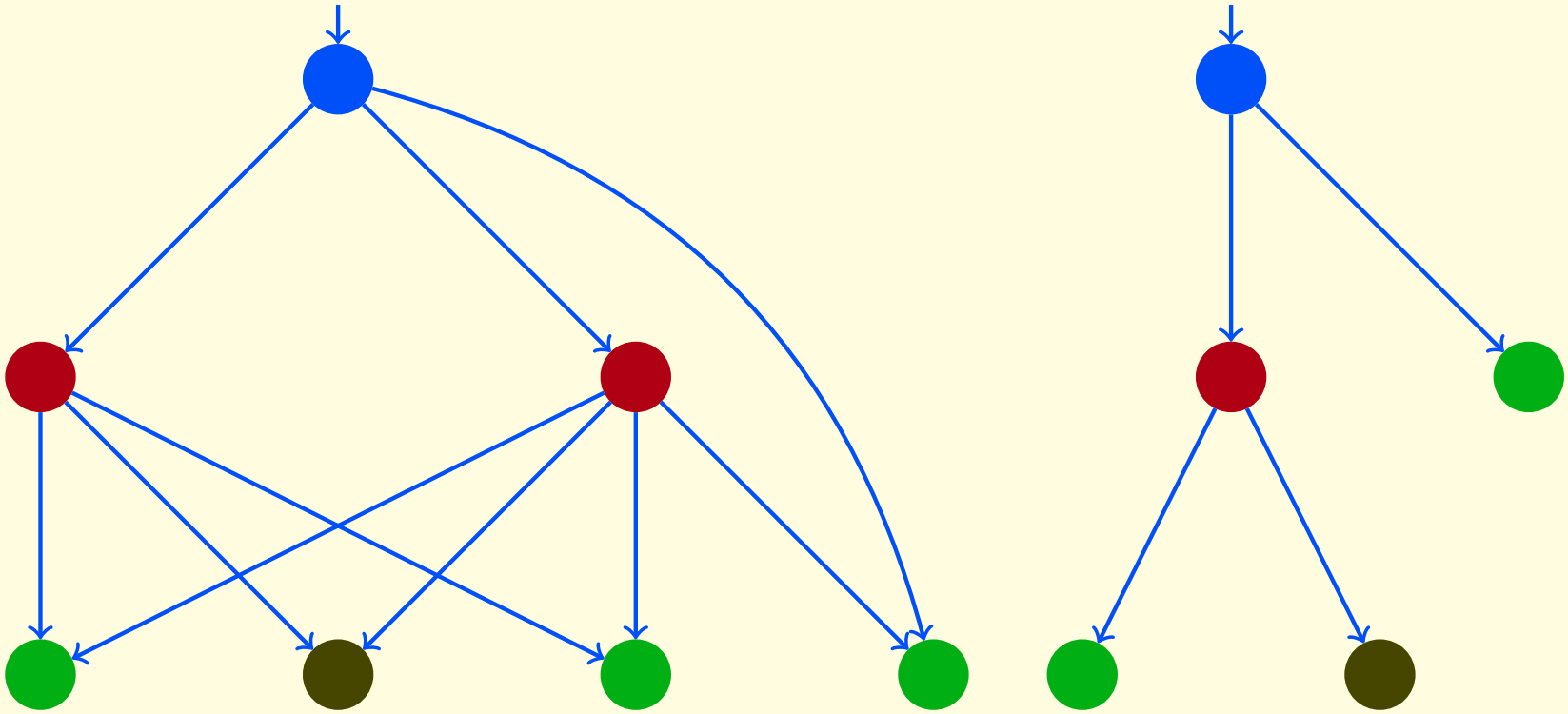


## Theorem (Reif)

*Player 0 wins from $v$ in $G^{imp}$ $\Leftrightarrow$ Player 0 wins from $\{v\}$ in $G^{perf}$.*

# Complexity Measures

Motivation:

- solve parity games in P despite imperfect information,
- but:
  - not known whether PARITY is in P
  - the powerset graph can be exponentially larger

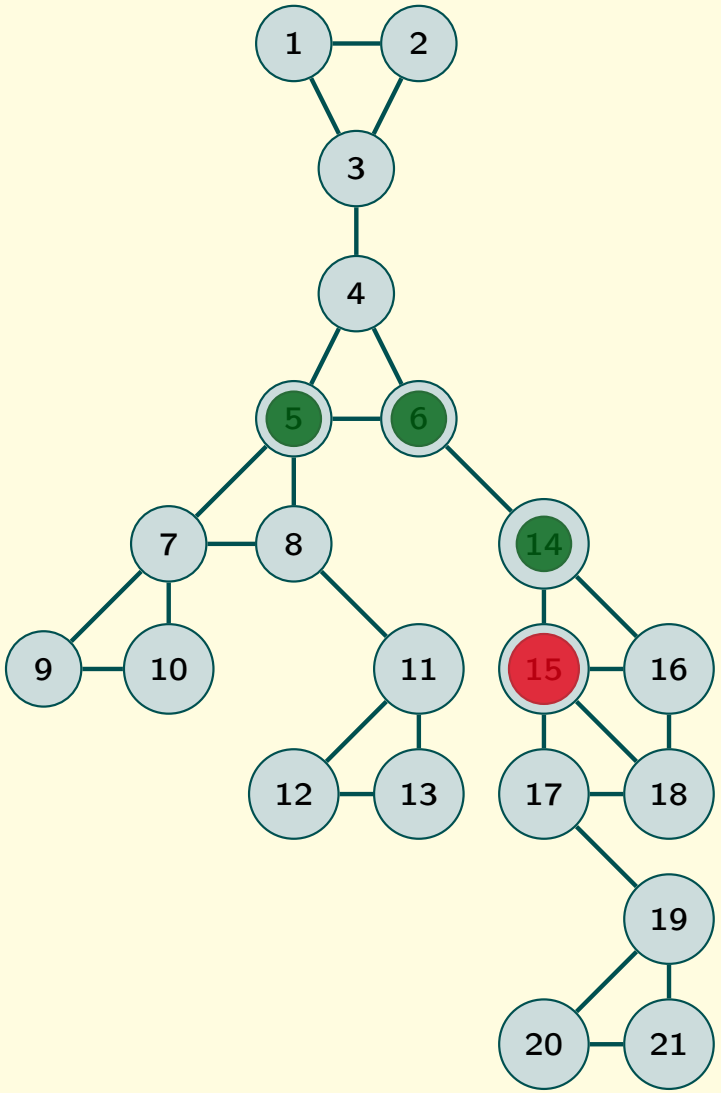
Hope: on simple graphs PARITY in P (like without imperfect information)

$\Rightarrow$ Need to measure complexity of a graph.

# Measures:

- **tree-width** (undirected graphs) $\oplus$ (Obdržálek 2003)
- directed tree-width ?
- **DAG-width** $\oplus$ (Berwanger et al. 2010)
- Kelly-width $\oplus$ (Kreutzer, Hunter 2008)
- entanglement $\oplus$ (Berwanger, Grädel 2005)
- . . . (not considered here)

# Tree-width: Game Theoretical Definition

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- $k$ Cops, one Robber
- Robber runs along cop free paths
- Cops fly
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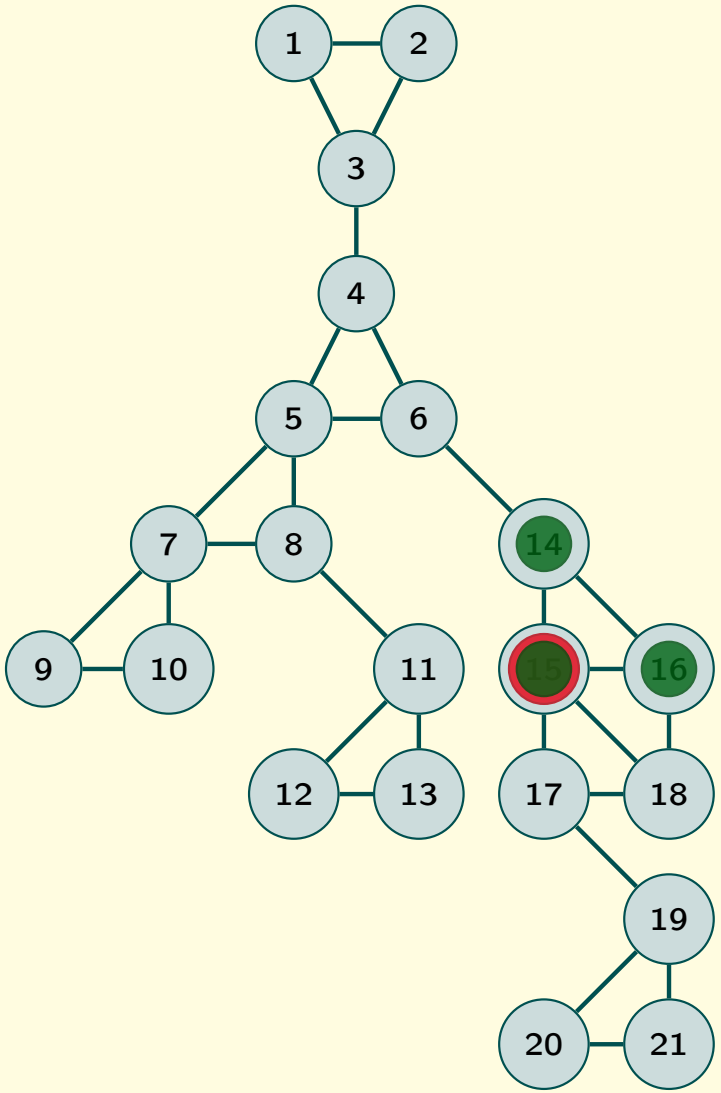


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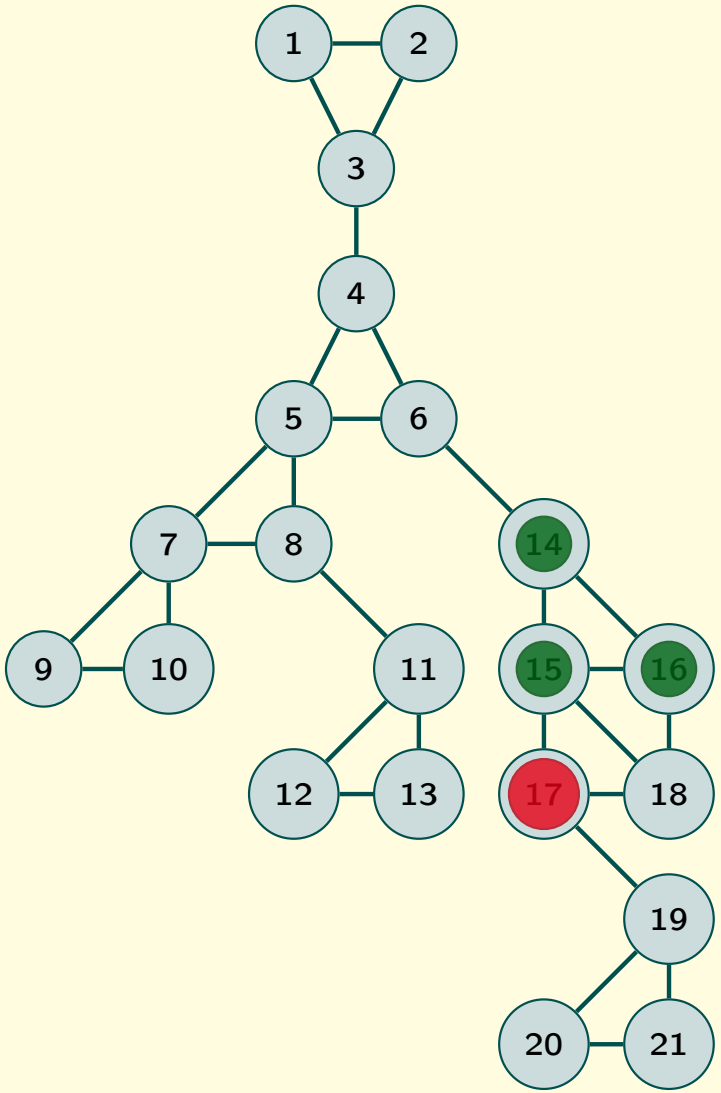


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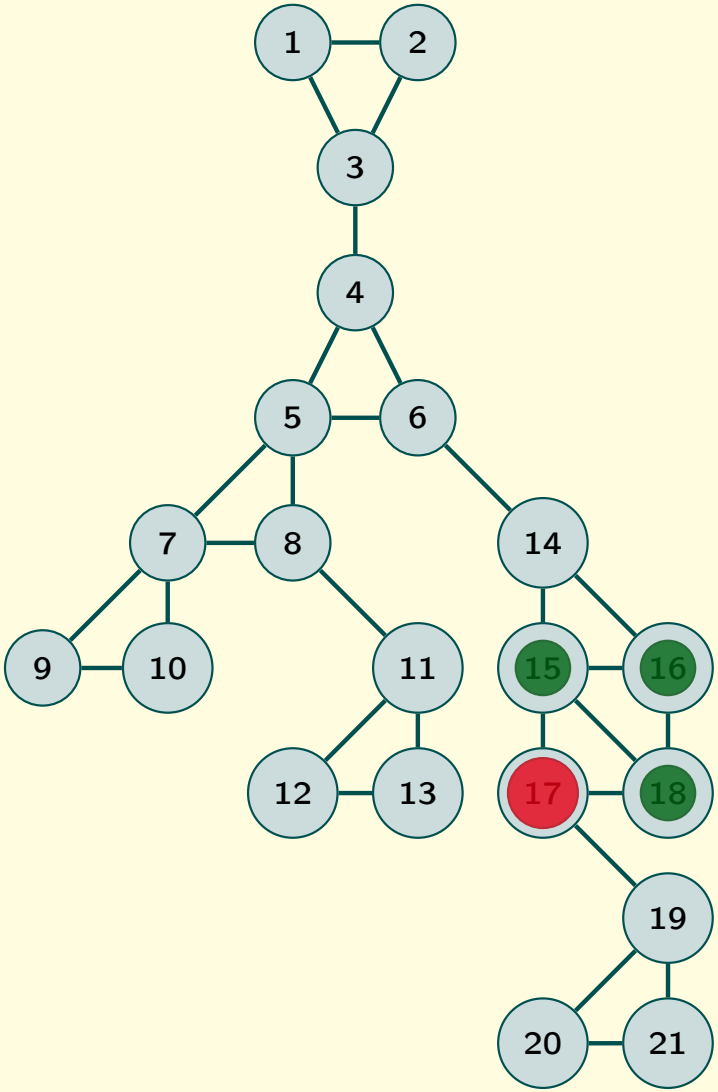


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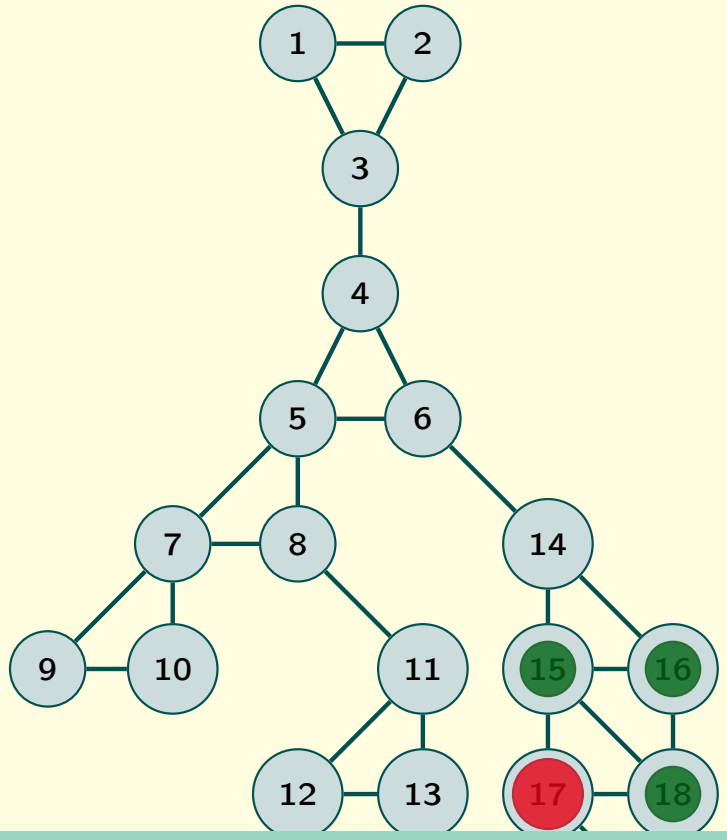


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Tree-width = minimal number cops monotonously capturing Robber − 1

# On Directed Graphs

- DAG-width game (Berwanger et al. 2010)
  - like before, but Robber runs along directed paths (+ monotonicity)
  - DAG-width bounded $\Rightarrow$ PARITY in P $\oplus$

# Unbounded Imperfect Information

*(General Case)*

# Measures Grow Exponentially

**Theorem**

*Exists $G^{imp}$ of small complexity, but $G^{perf}$ of exponential complexity (w.r.t. all our measures).*

- very large information sets

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*Reachability: EXPTIME-hard even if entanglement $\leq 2$ and directed path-width $\leq 2$. (Based on original idea for hardness by Reif.)*

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- further restrictions needed
- natural approach: bound size of information sets

# Bounded Imperfect Information

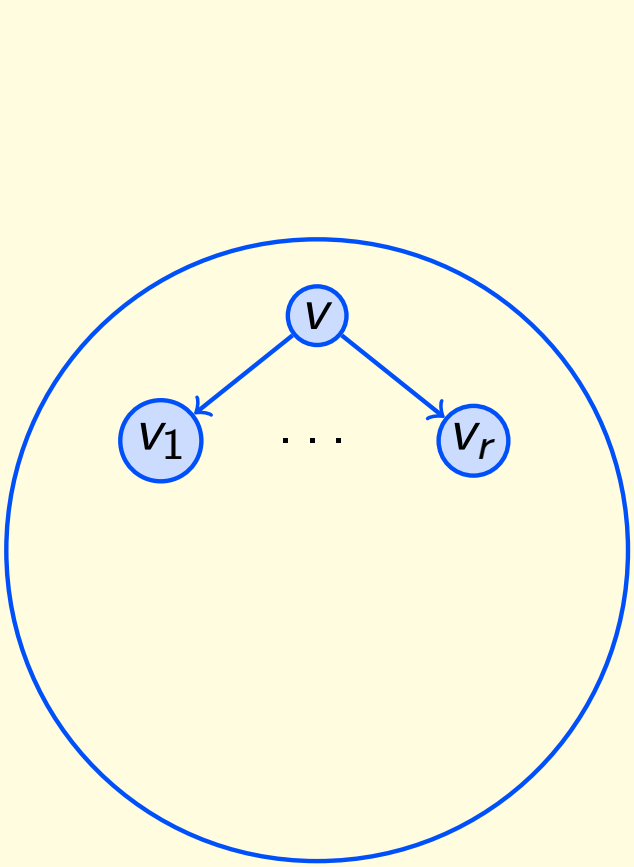
# General Procedure

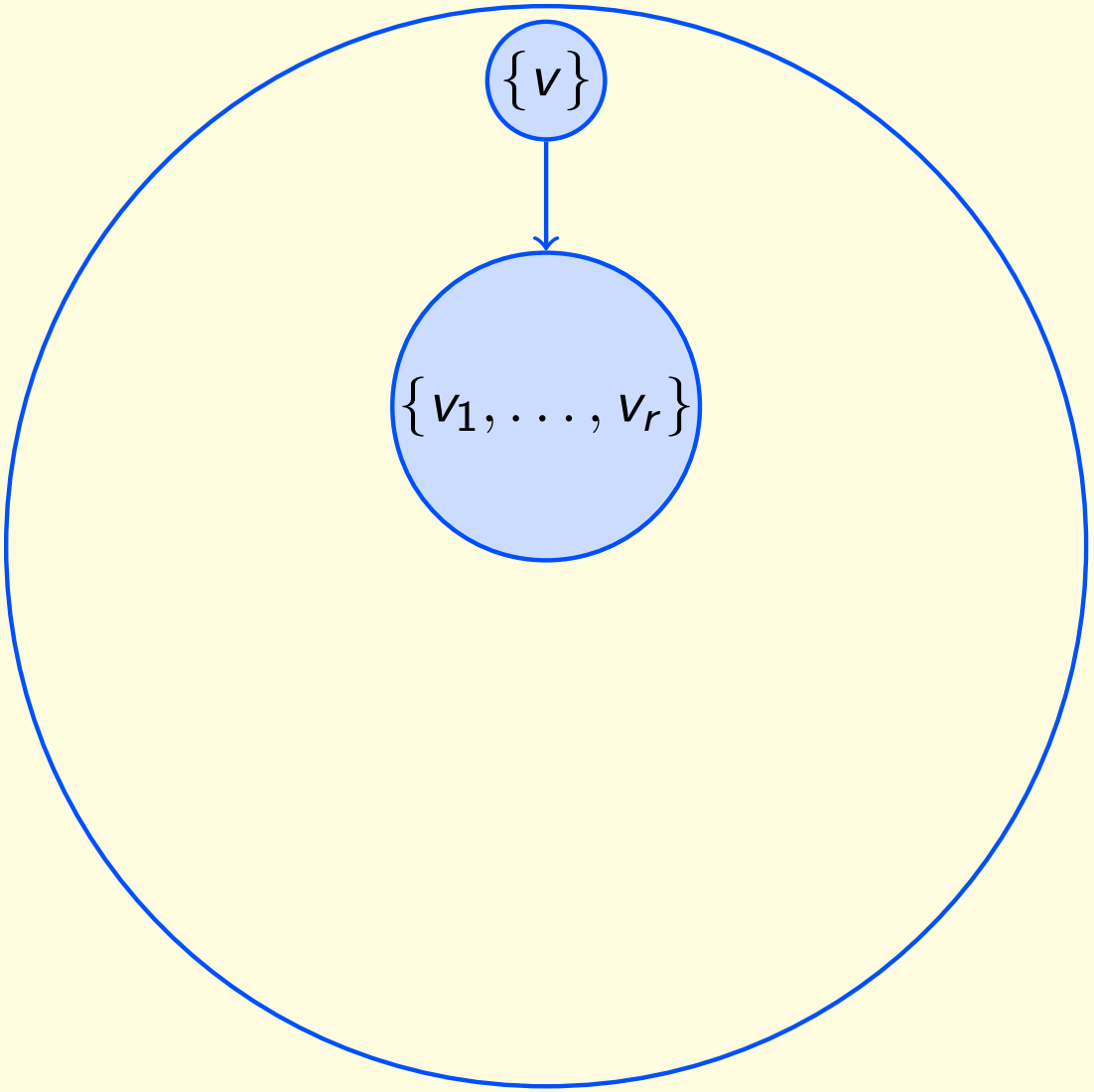
- show for appropriate $\oplus$ measures:
  if measure($G^{imp}$) $\leq k$, |information sets| $\leq r$
  $\Rightarrow$ measure($G^{perf}$) $\leq f(k, r)$
- then
  if measure($G^{imp}$) $\leq k$ and |information sets| $\leq r$
  ($\Rightarrow |G^{perf}|$ polynomial)
  $\Rightarrow$ PARITY in P

# How Measures Behave

- tree-width: $\ominus$ $\mathrm{tw}(G^{imp}) = 2$, but $\mathrm{tw}(G^{perf})$ unbounded
  - **still:** $\oplus$ $\mathrm{tw}(G^{imp})$ bounded $\Rightarrow$ DAG($G^{perf}$) bounded
- entanglement: $\ominus$ $\mathrm{ent}(G^{imp}) = 2$, but $\mathrm{ent}(G^{perf})$ unbounded
- DAG-width:
  - *non-monotone* (not appropriate): $\oplus$ $f(k, r) = k \cdot r \cdot 2^{r-1}$
  - if every information set is an SCC: $\oplus$, $f(k, r) = k \cdot r^2 \cdot 2^{r-1}$
  - if every |information set| $\leq 2$: $\oplus$, $f(k, r)$ bounded
  - **in general:** $\oplus$ a newer result: $f(k, r) = 2kr \cdot 2^{r-1}$
- Kelly-width: ?
- directed tree-width: ? (idea doesn't work)

# General Proof Idea for Boundedness of Measures



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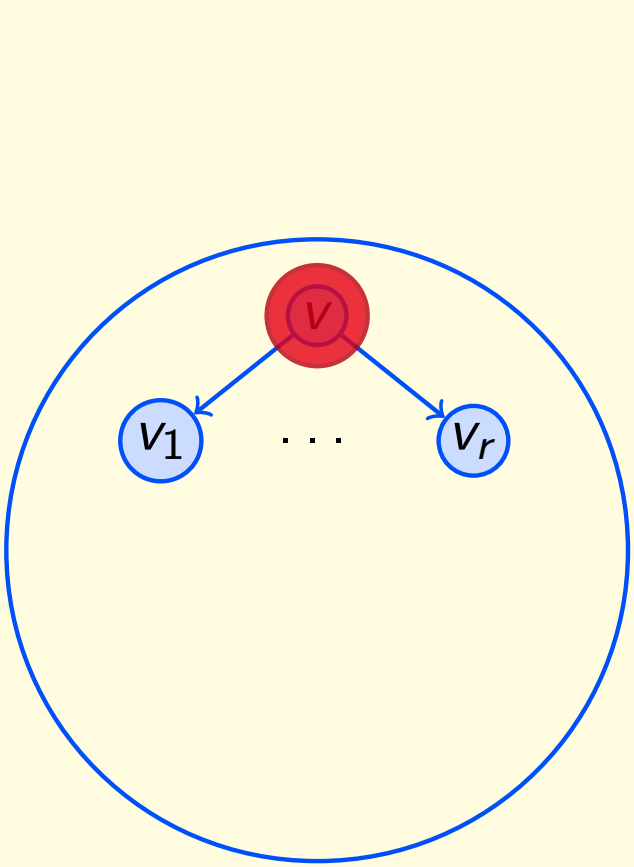


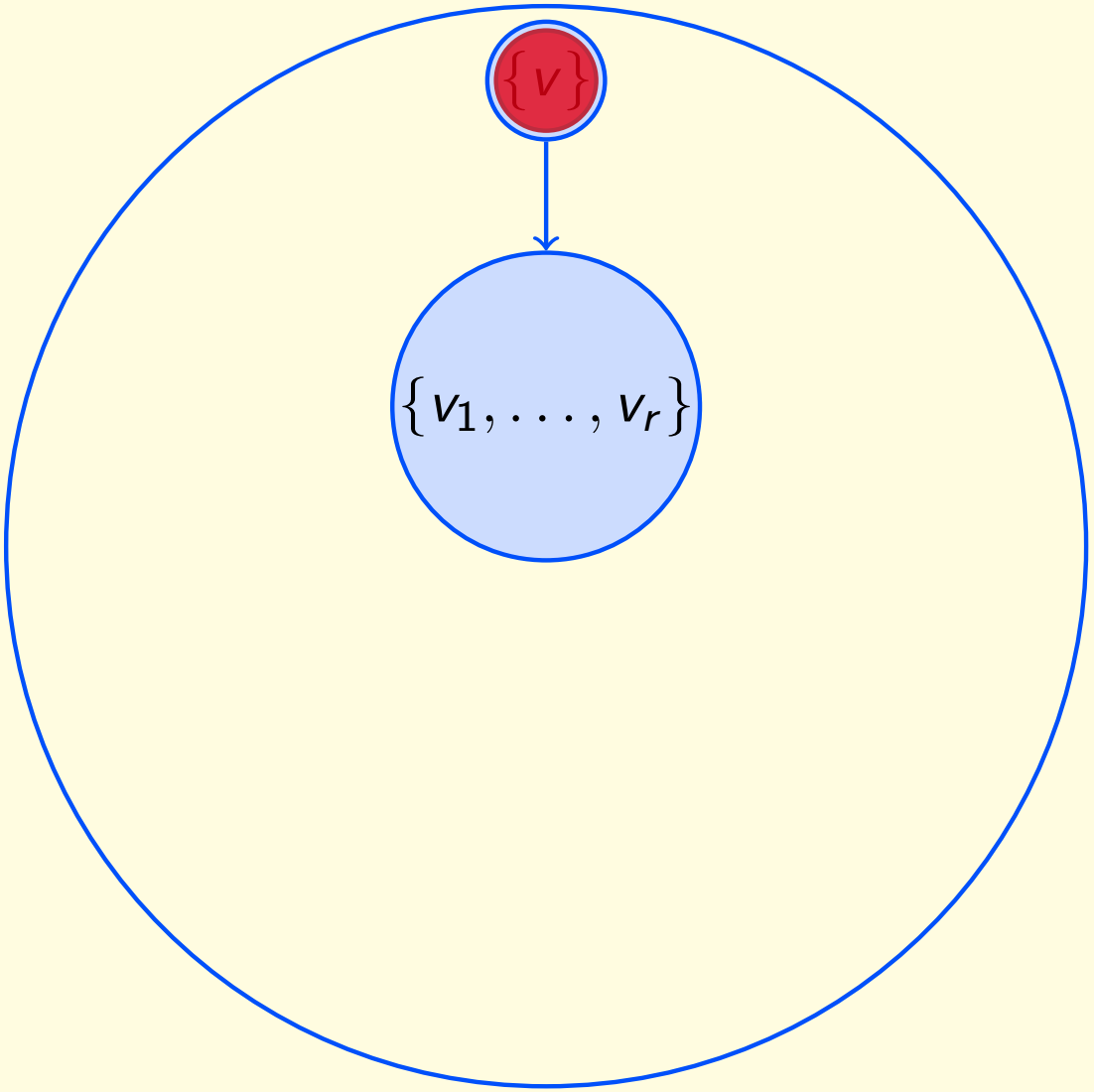

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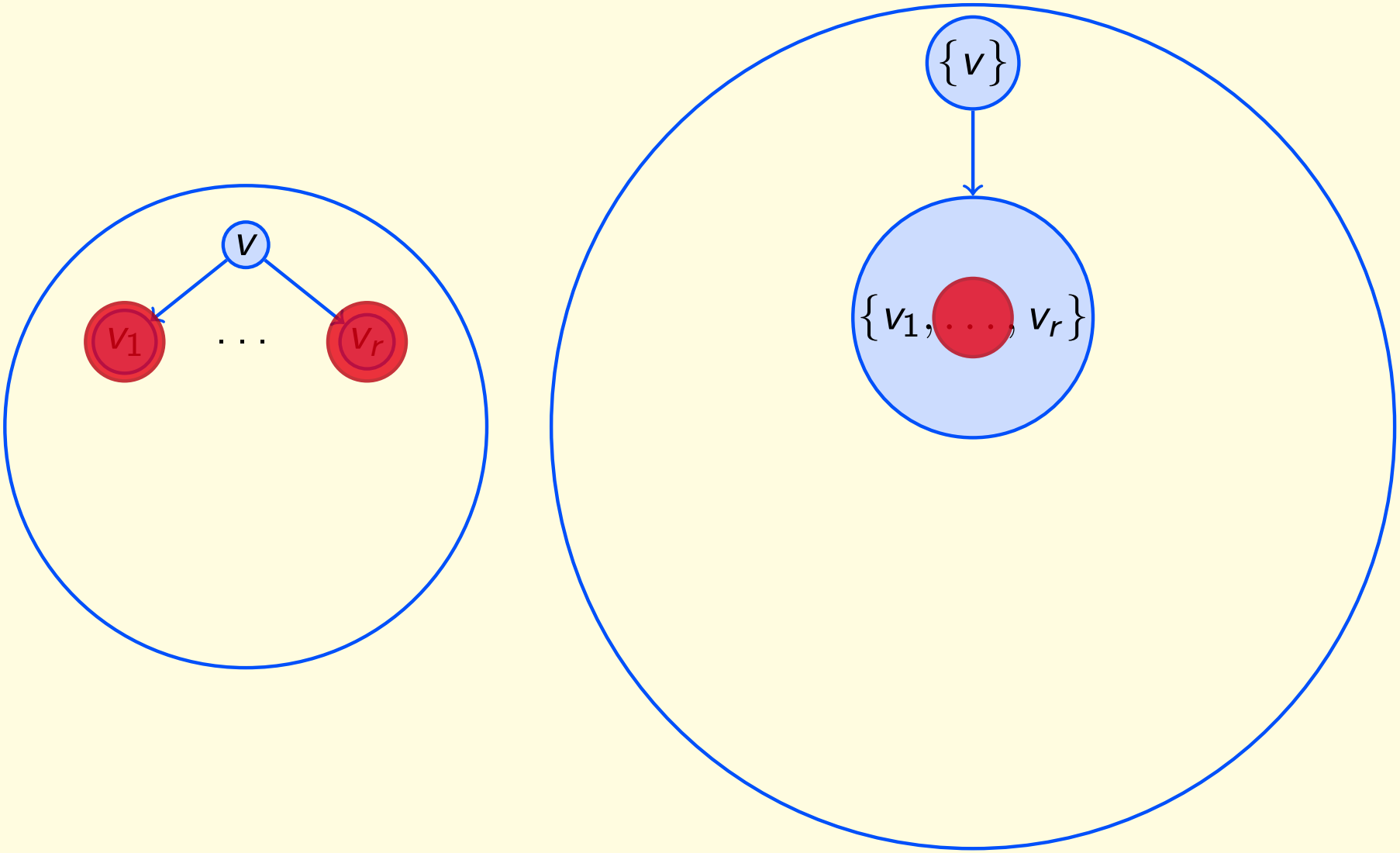

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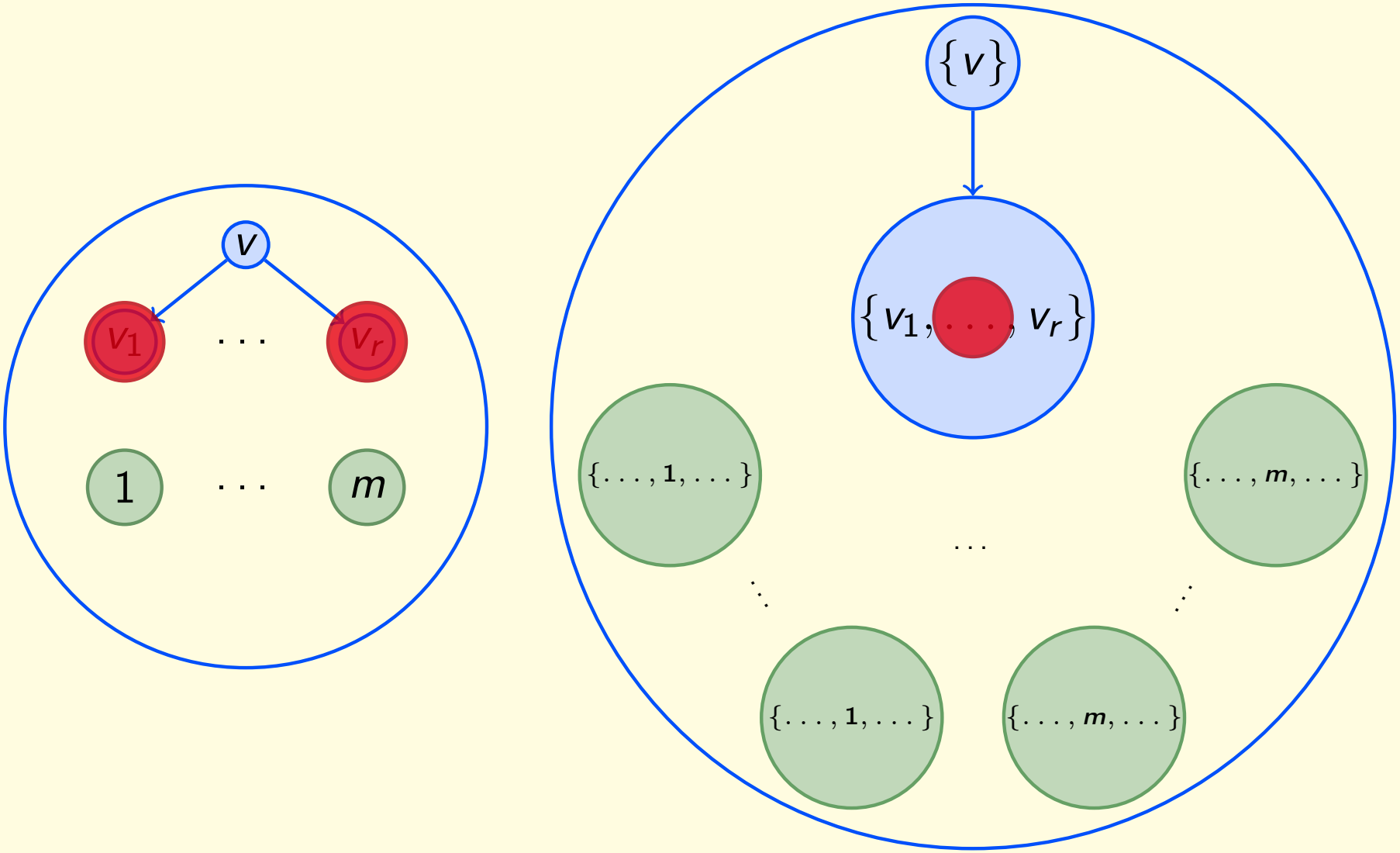

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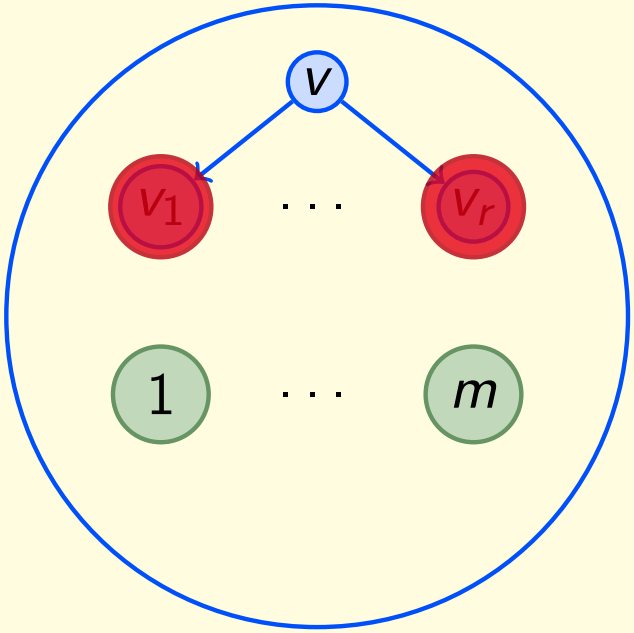
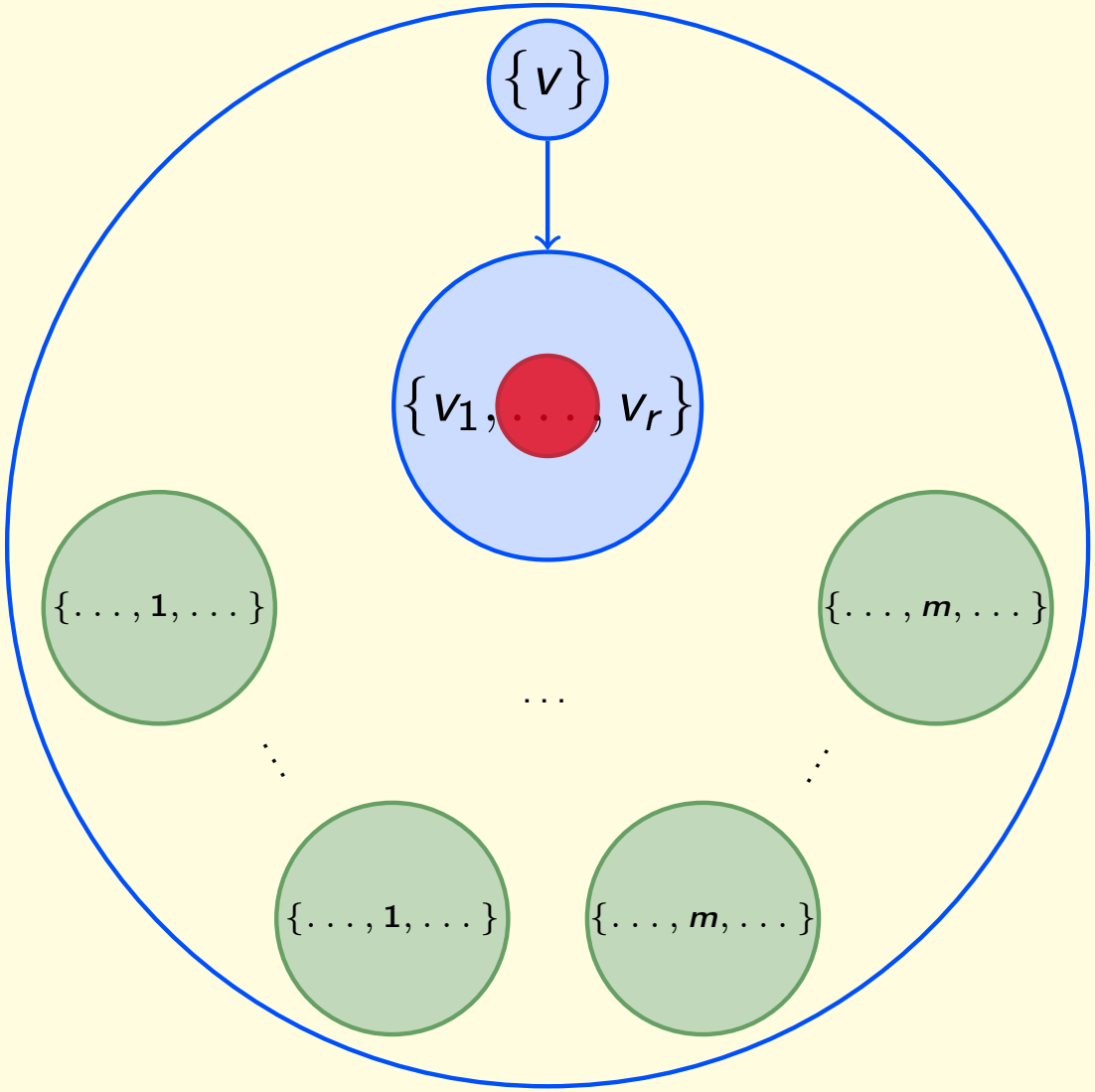
nm-DAG-width:

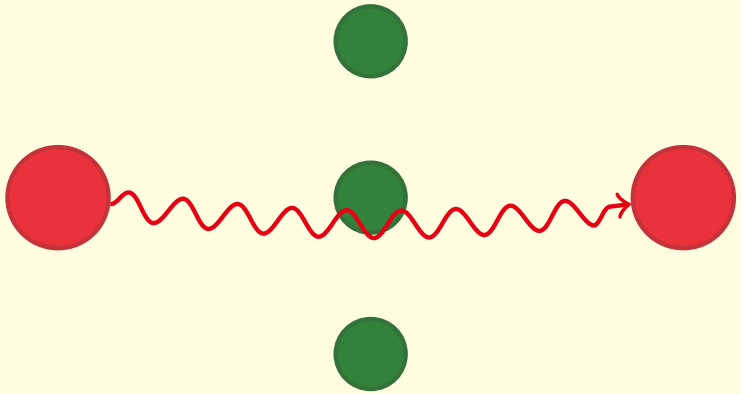
$f(k, r) = k \cdot r \cdot 2^{r-1}$



# New DAG-Game: Multiple Robbers

- Cops must capture $r$ Robbers.
- Robbers can jump:



- Monotonicity:
  no robber can access a vertex that was inaccessible to all robbers

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$k$ cops win Tree-width Game

$\Leftrightarrow$ forget directions of edges and win

Stronger assumption: robber has more paths to run!

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## Theorem

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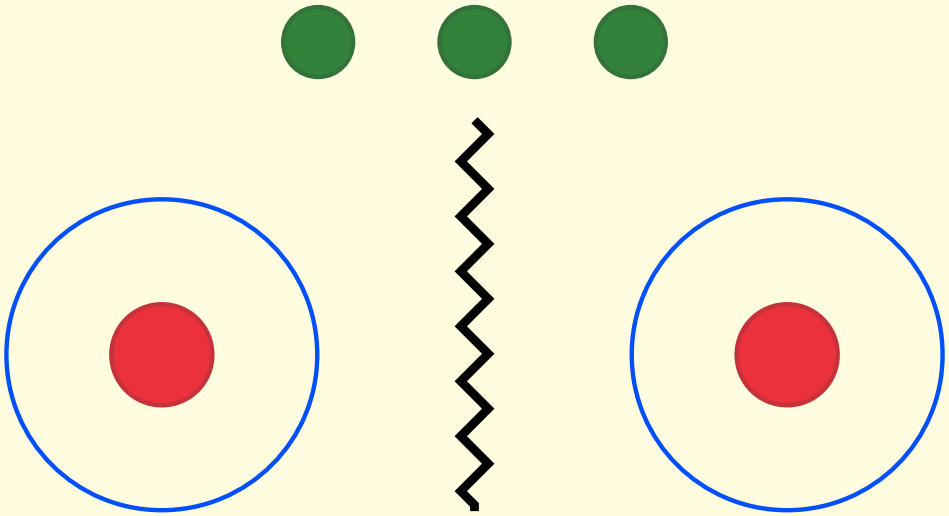
*If $k$ cops monotonously win against one robber*

*then $k \cdot r$ cops monotonously win against $r$ robbers.*



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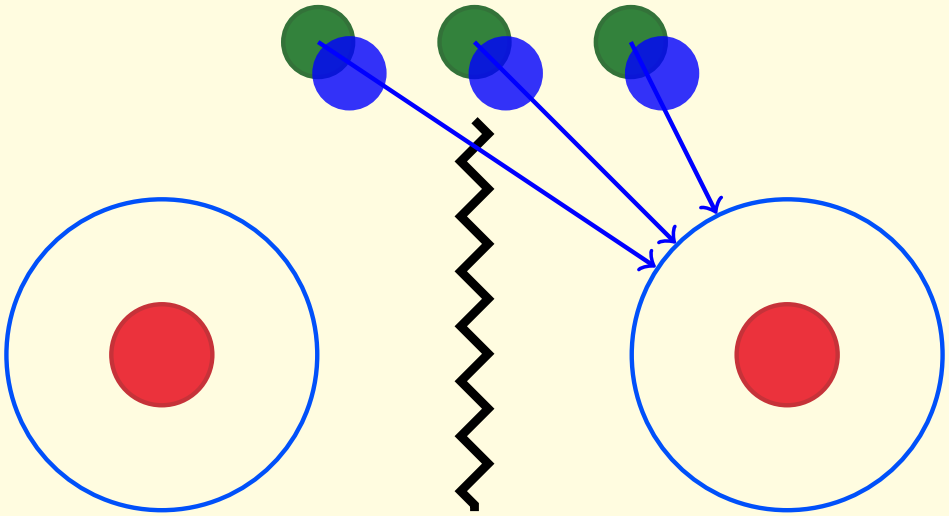

# Summing up for Tree-width

**Theorem**

*If $|$information set$| \leq r$ and tree-width of $G \leq k$ then PARITY is efficiently solvable on $G$.*

What doesn't work for DAG-width?

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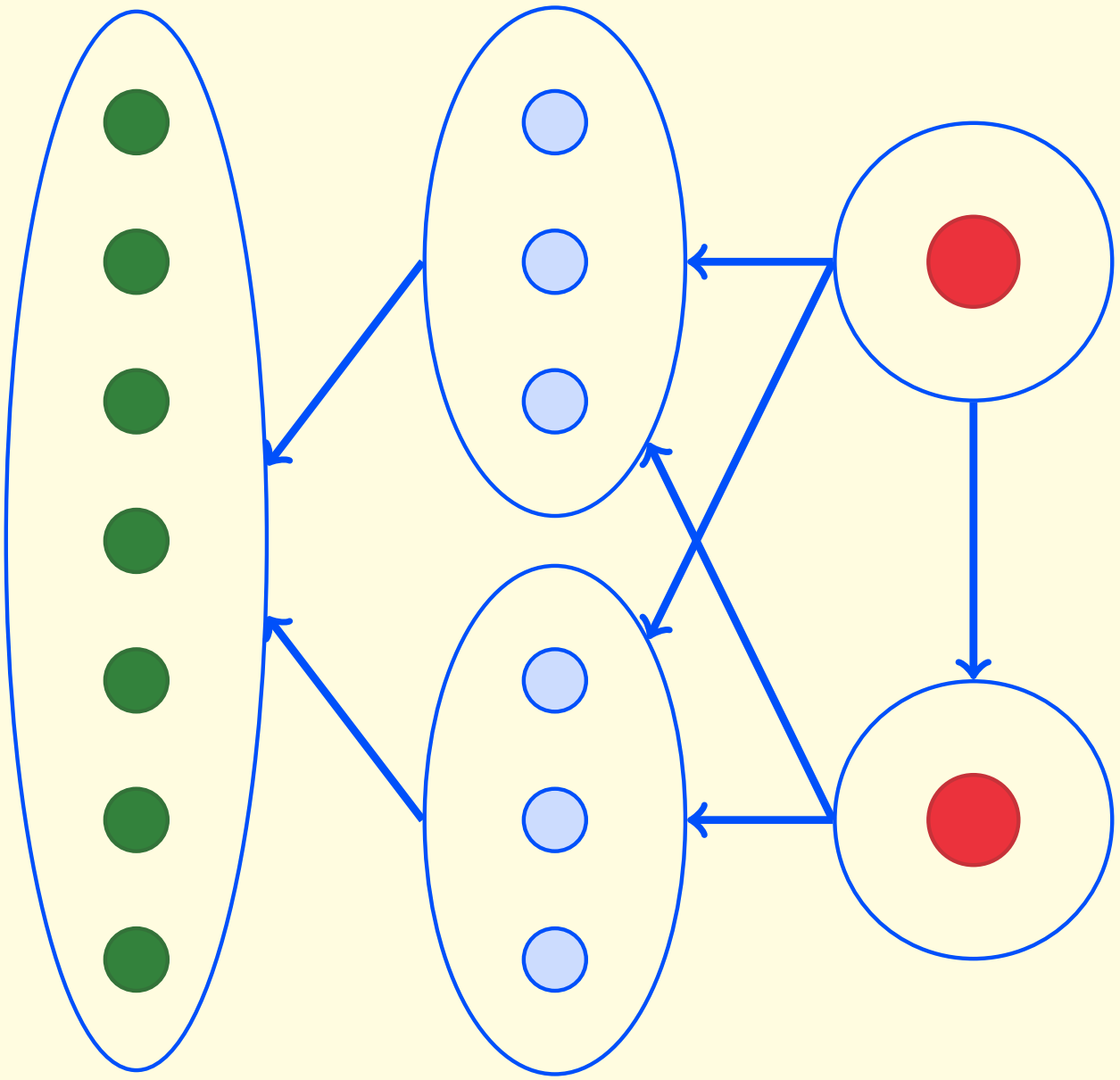

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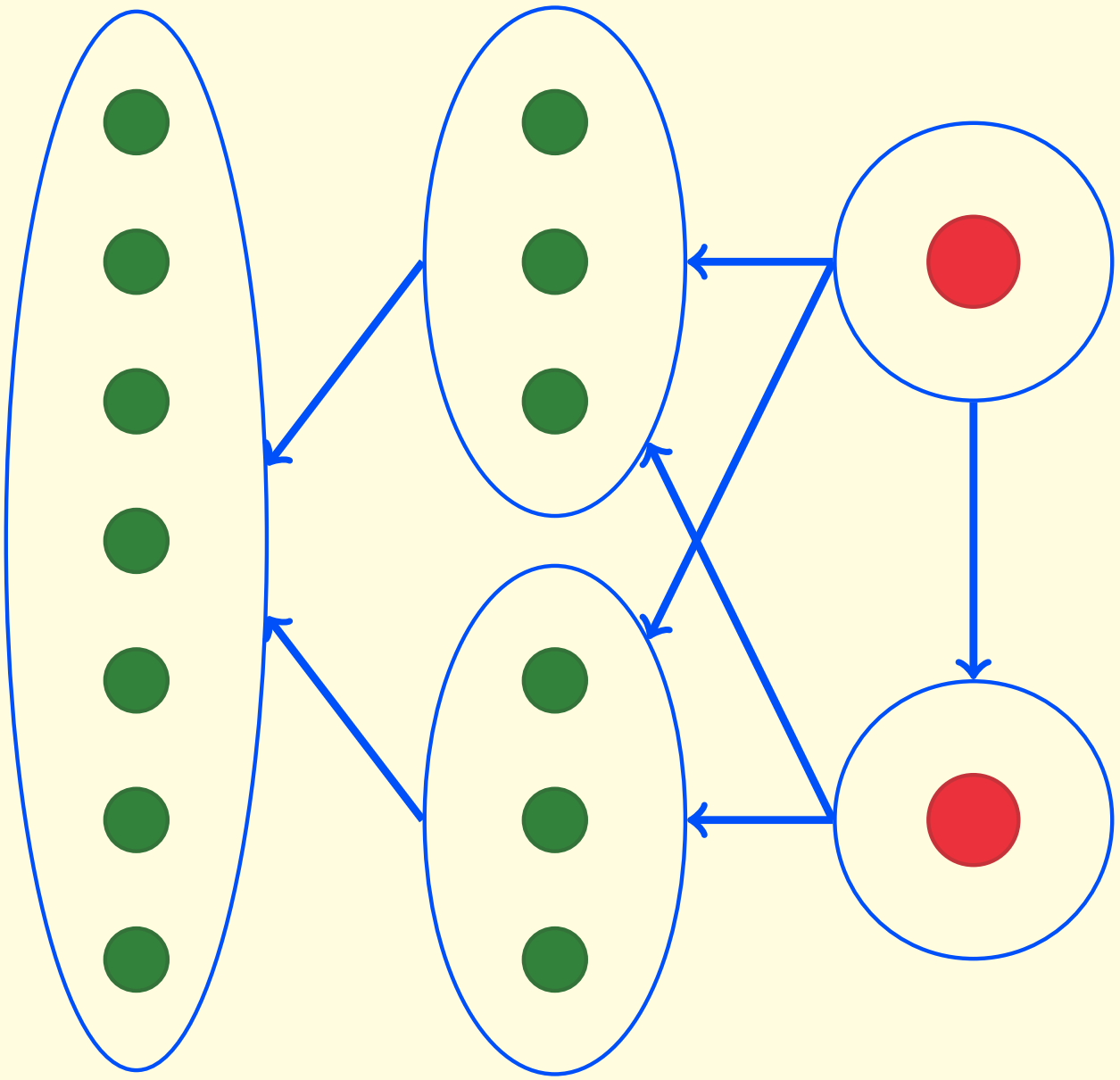

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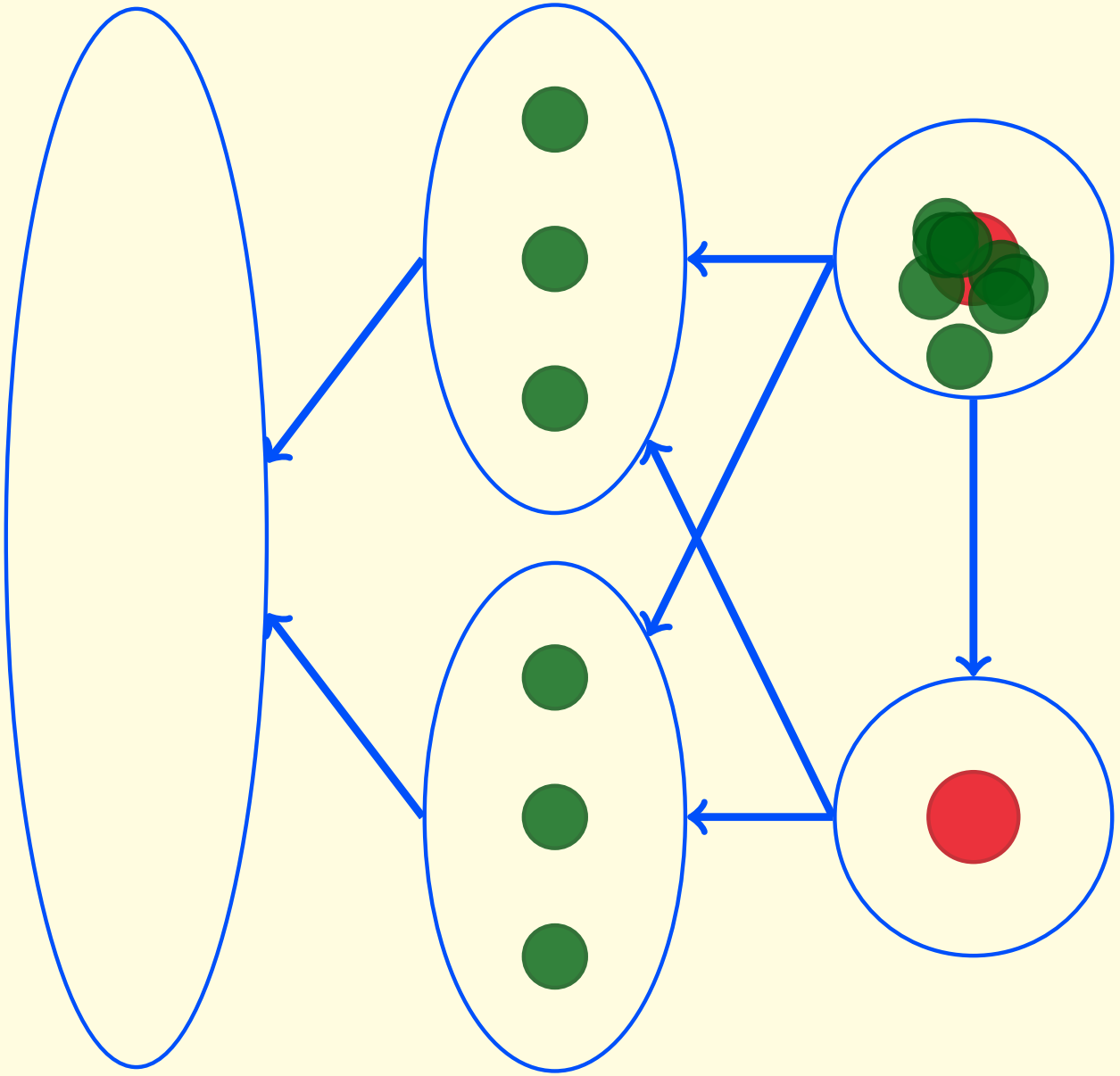

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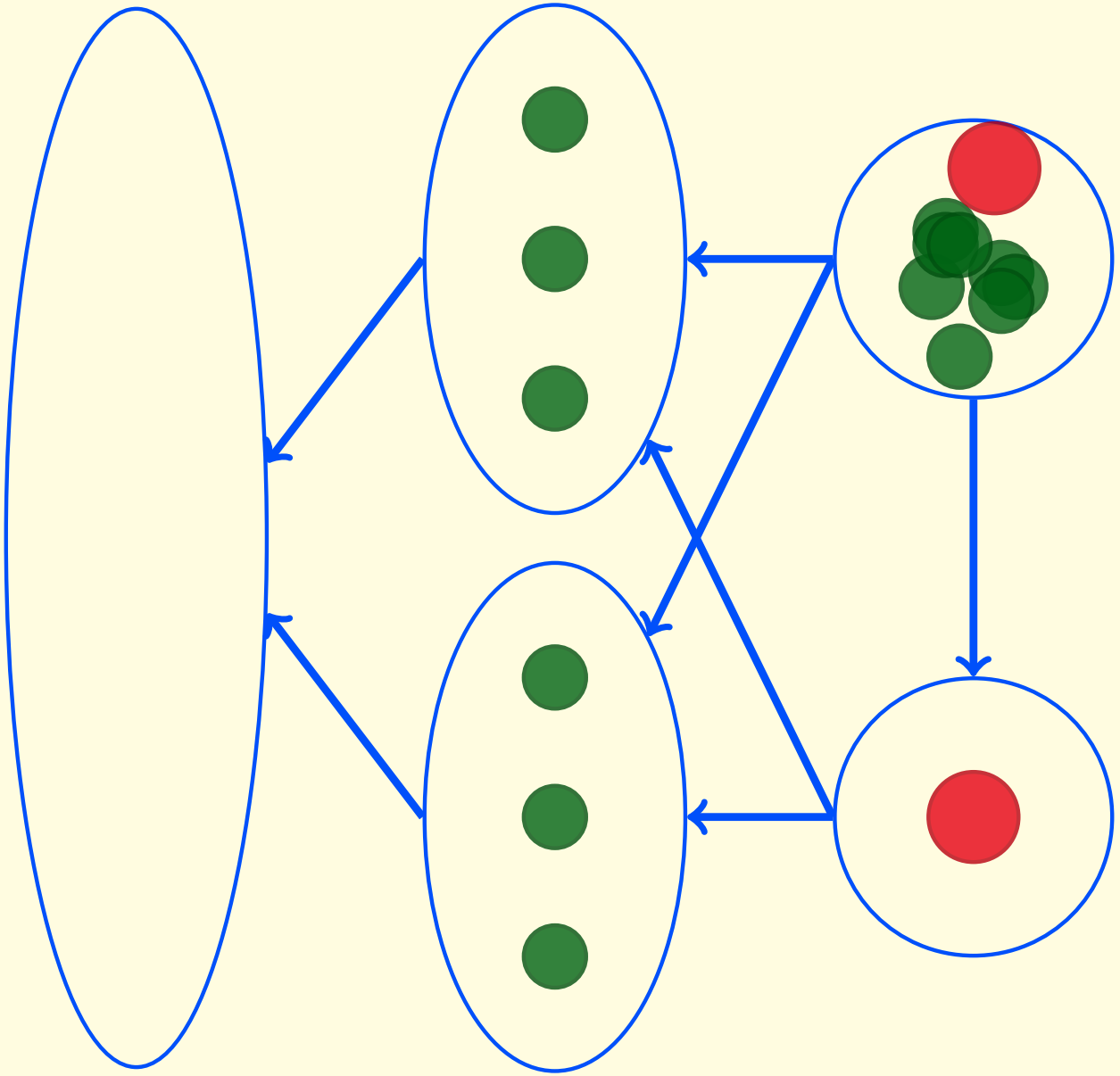

# Place Cops only Inside?

## Theorem (Kaiser, Puchala, R.)

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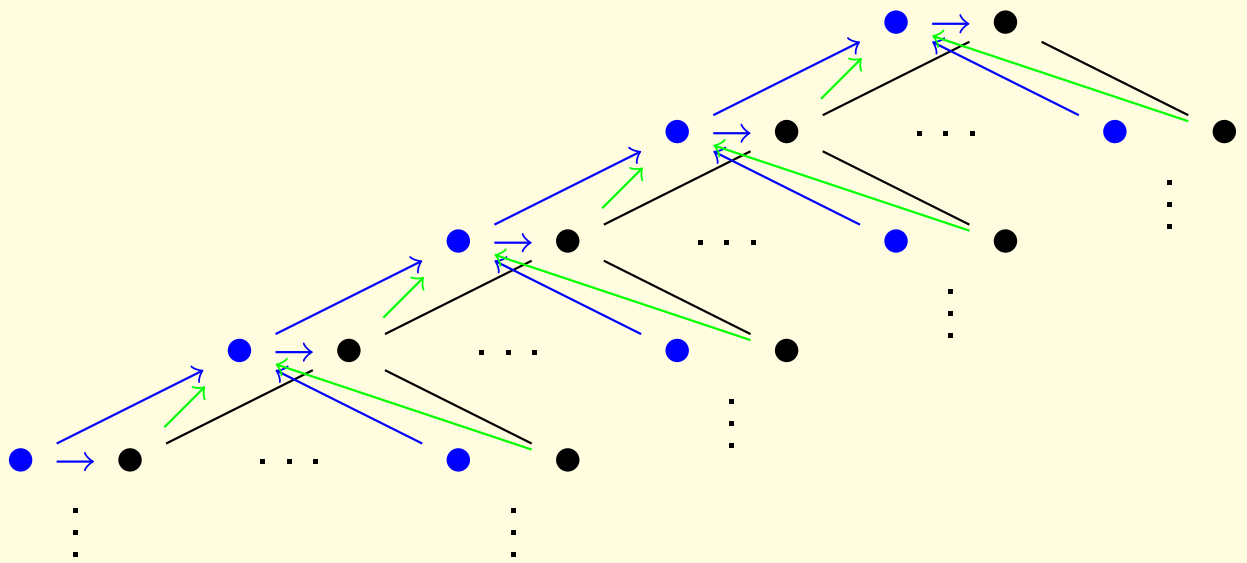
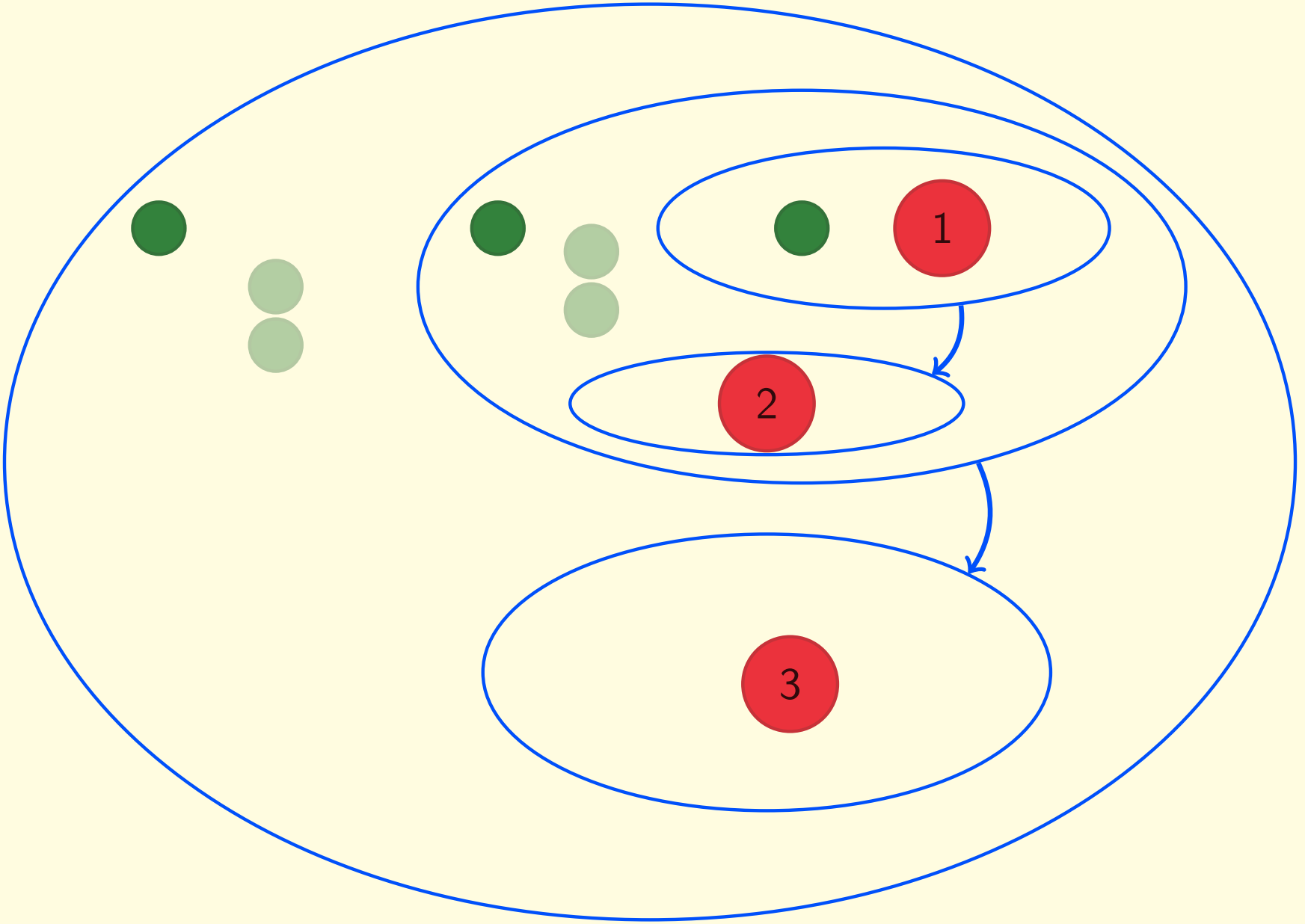


# Solving Monotonicity Problem for $r$ Robbers



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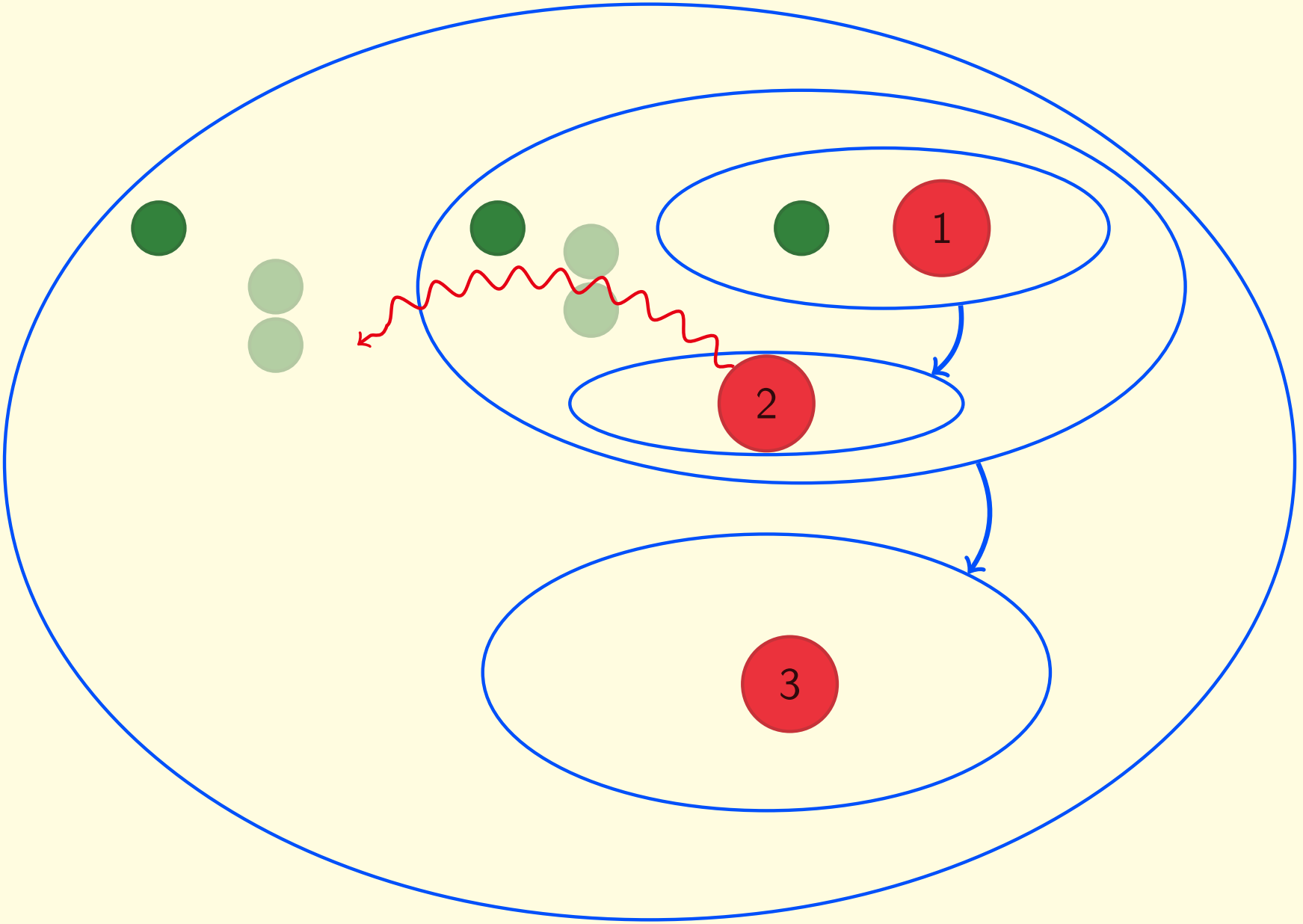

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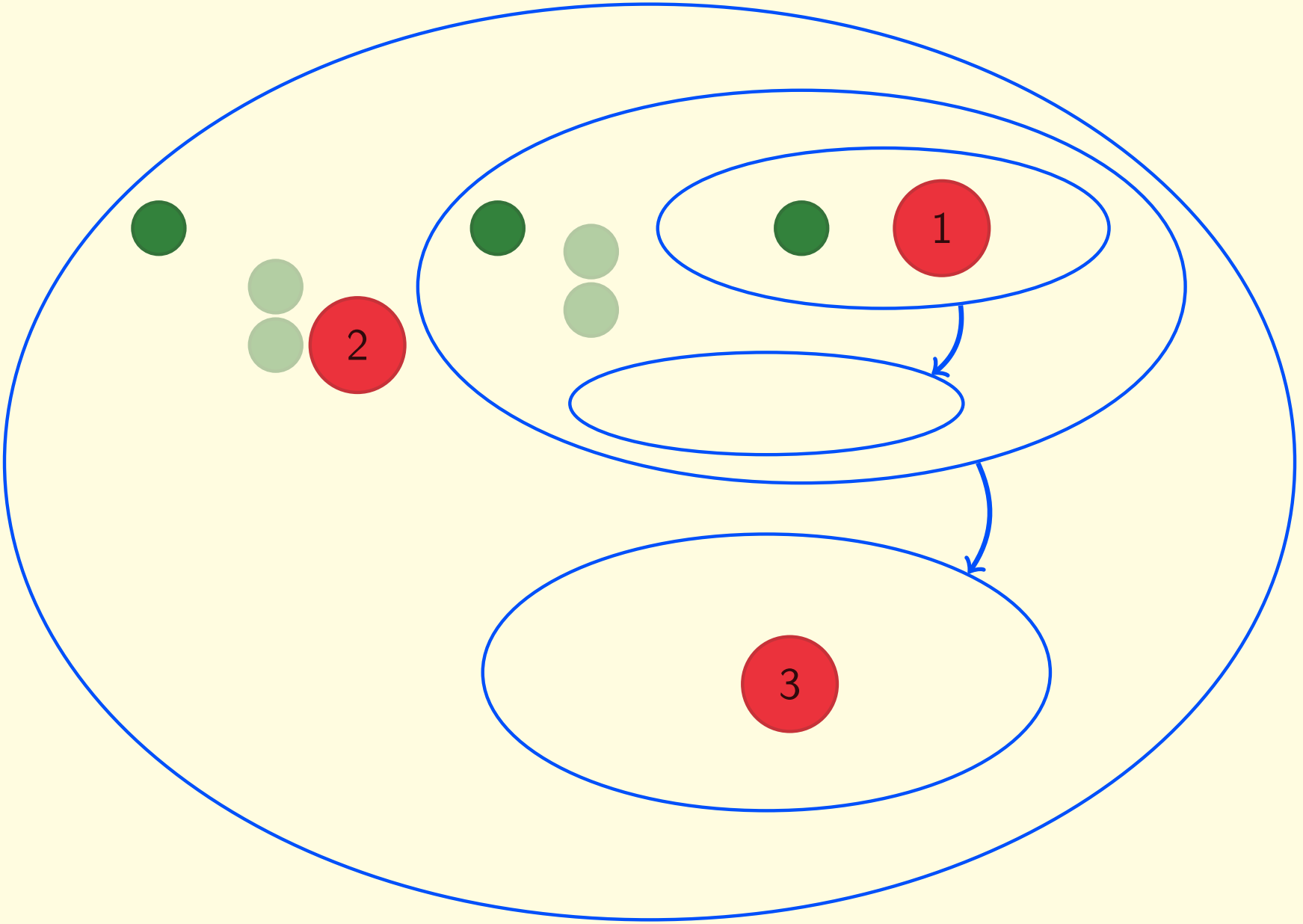

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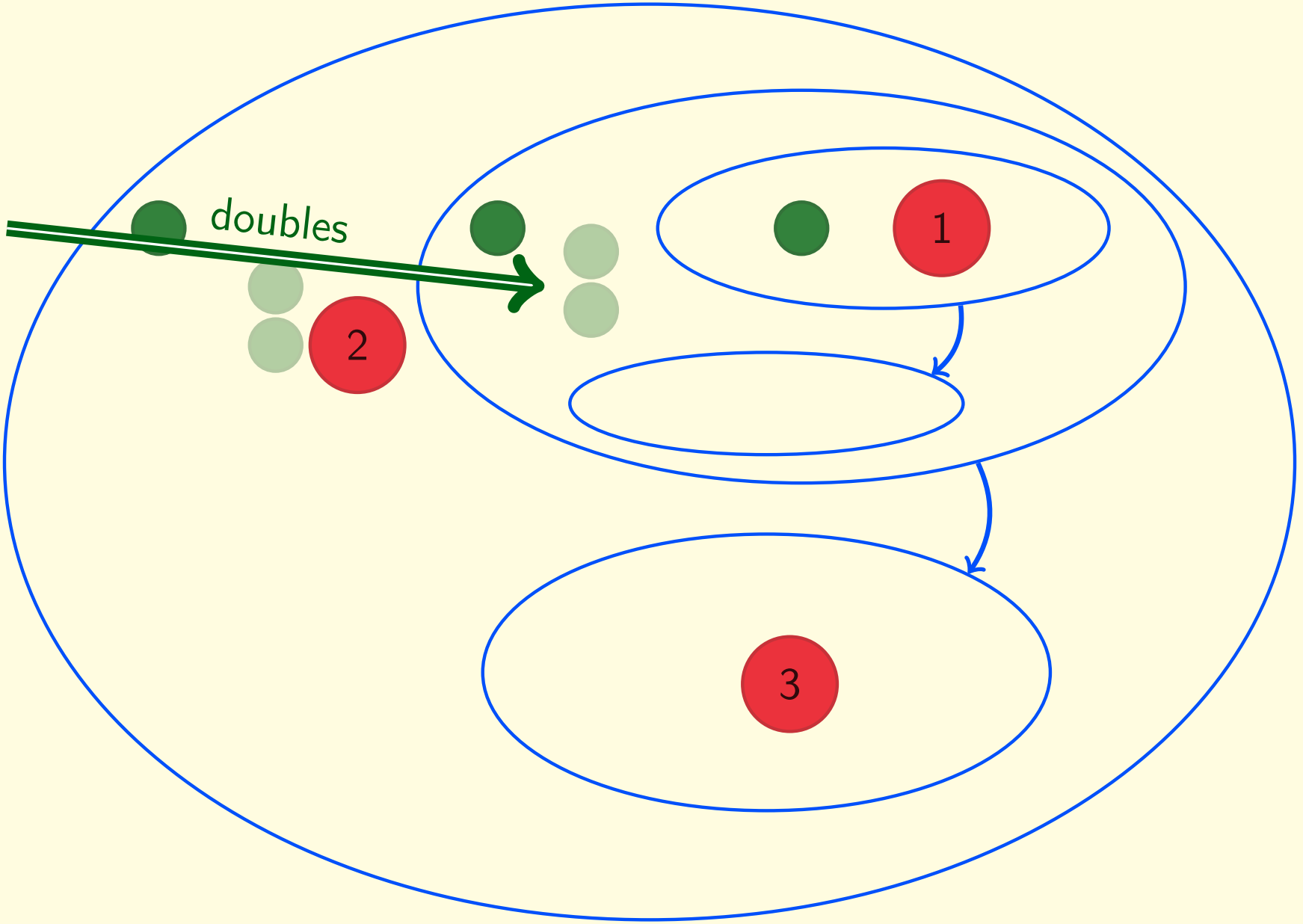

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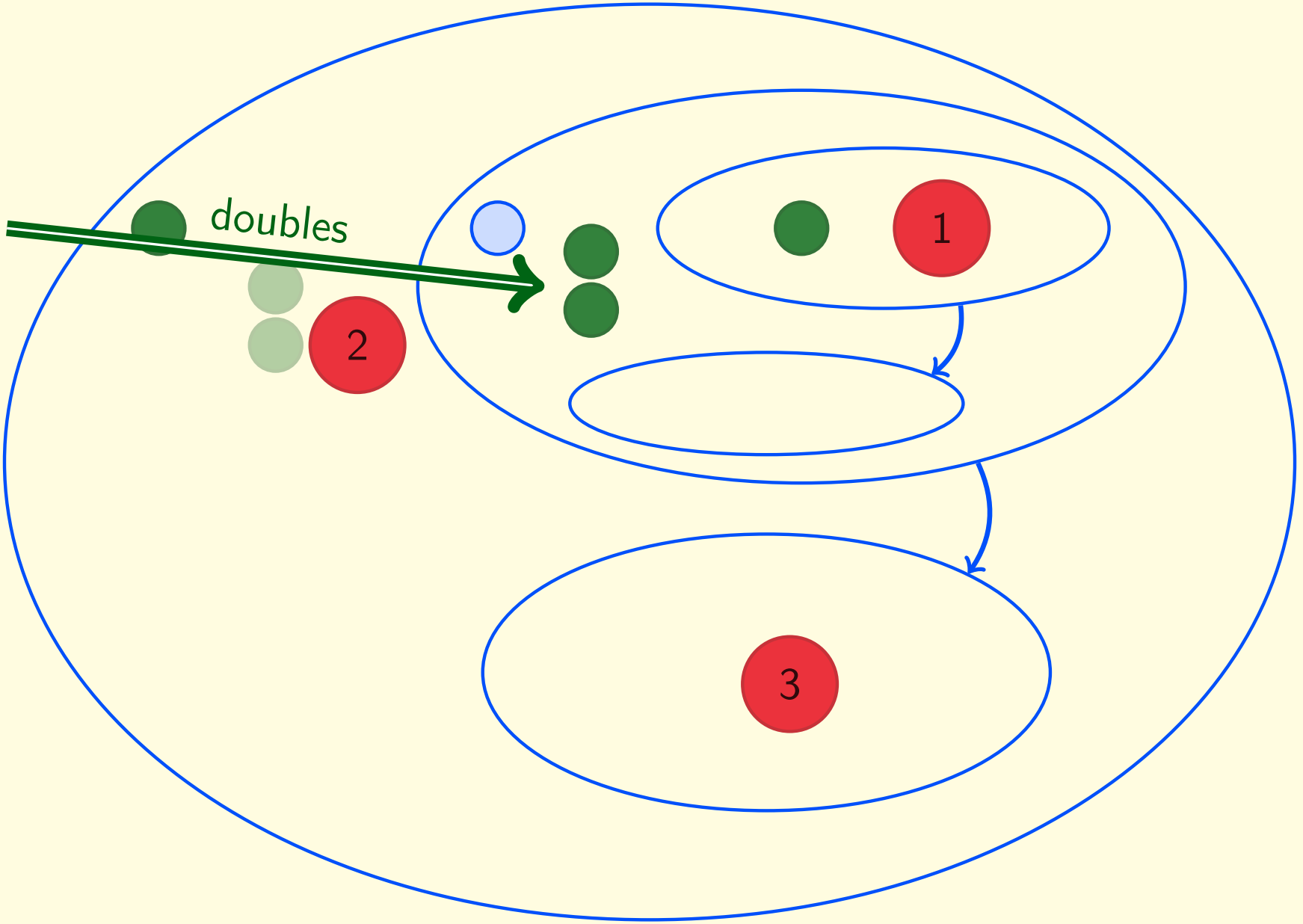

Thank you for your attention!