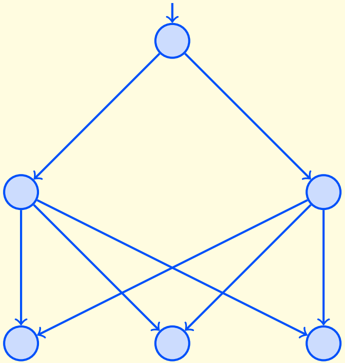
Parity Games with Imperfect Information on Graphs of Bounded Complexity

Bernd Puchala Roman Rabinovich

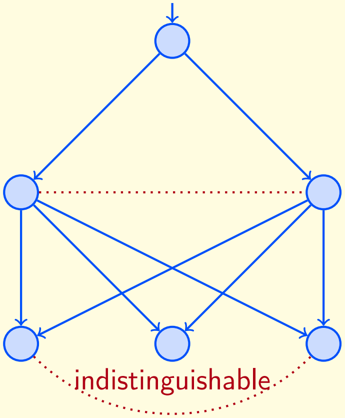
RWTH Aachen University

May 2010

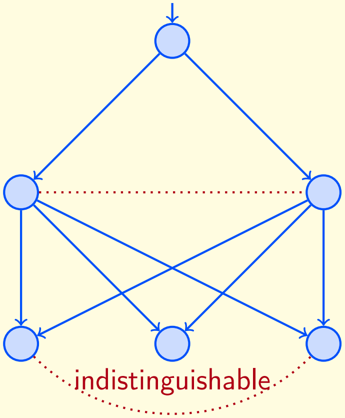
Imperfect Information



Imperfect Information

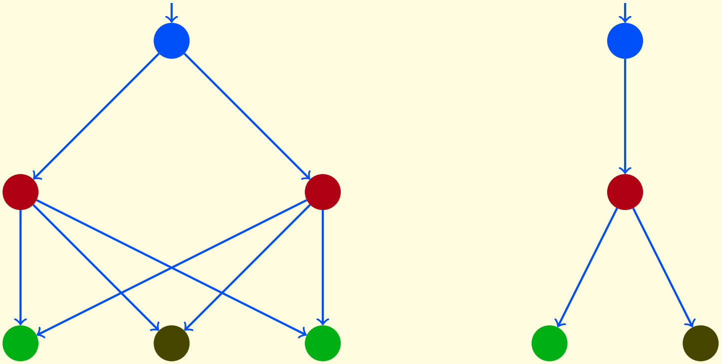


Imperfect Information

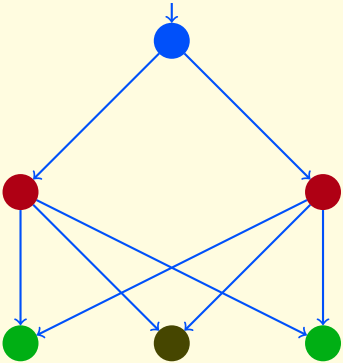


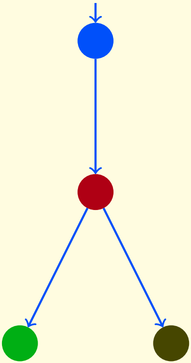
- Indistinguishable vertices build an information set.
- Add colours to the vertices (for parity condition).
- Technical subtleties:
  - Edges are labeled.
  - Edge labels may be indistinguishable.
  - Edge labels compatible with information sets.
  - Colours compatible with information sets.
  - Start vertex distinguishable from all.
  - Mark vertices of Player 0.

Powerset Construction



Powerset Construction

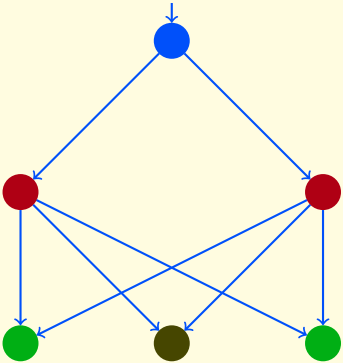


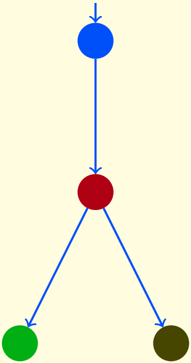


Theorem (Reif)

Player 0 wins from v in G^{imperf} $\Leftrightarrow$ Player 0 wins from $\{v\}$ in G^{perf}

Powerset Construction

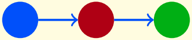
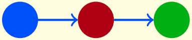




Theorem (Reif)

Player 0 wins from v in G^{imperf} $\Leftrightarrow$ Player 0 wins from $\{v\}$ in G^{perf}

Lemma

Path  *in G^{imperf} $\Leftrightarrow$ path*  *in G^{perf}.*

Complexity Measures

Motivation:

- solve parity games in P despite imperfect information,
- but:
  - not known whether PARITY is in P
  - the powerset graph can be exponentially bigger

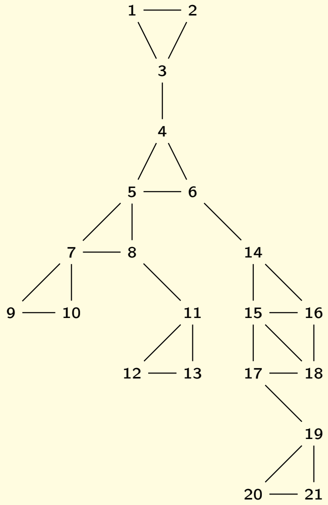
Hope: on simple graphs PARITY in P (like without imperfect information)

$\Rightarrow$ Need to measure complexity of a graph.

Measures:

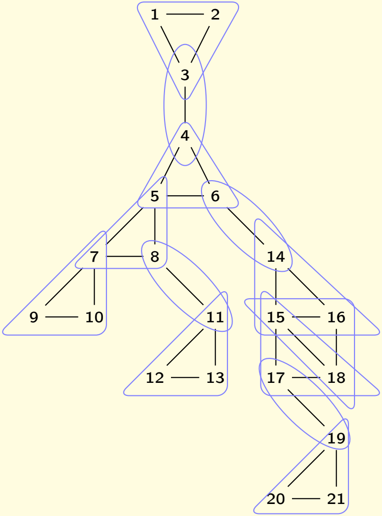
- tree-width (undirected graphs) + path-width $\oplus$
- directed tree-width ?
- DAG-width + directed path-width $\oplus$
- Kelly-width $\oplus$
- entanglement $\oplus$
- . . . (not considered here)

Tree-width: Tree Decomposition



simple = like a tree
complex = like a grid

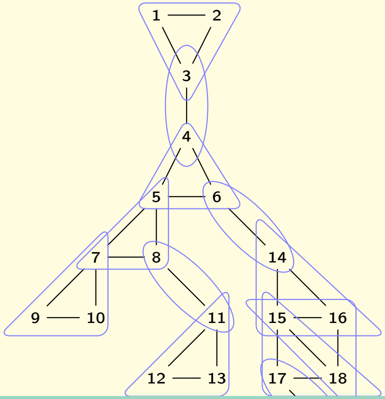
Tree-width: Tree Decomposition



simple = like a tree
complex = like a grid
Bags:

- treelike connected
- contain every vertex
- contain every edge
- bags with vertex *v* induce one subtree

Tree-width: Tree Decomposition

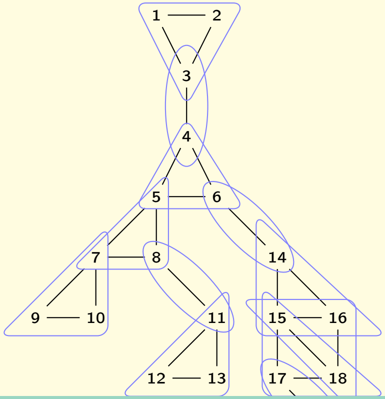


simple = like a tree
complex = like a grid
Bags:

- treelike connected
- contain every vertex
- contain every edge
- bags with vertex v induce one subtree

Tree-width: minimal size of the greatest bag -1

Tree-width: Tree Decomposition



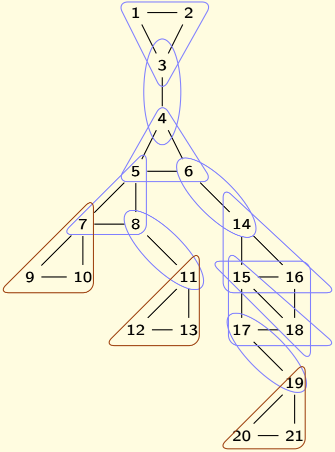
simple = like a tree
complex = like a grid
Bags:

- treelike connected
- contain every vertex
- contain every edge
- bags with vertex v induce one subtree

Path-width: the same, but path instead of tree

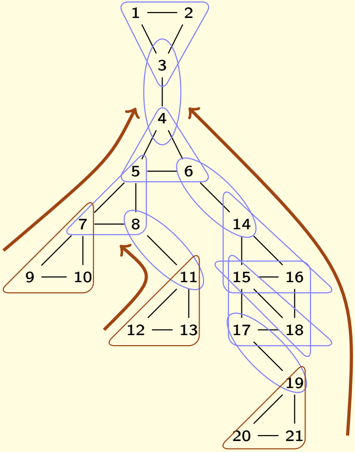
Dynamic Programming on Tree Decomposition

- small bags ⇒ arbitrary computations on leaf bags
- use: connections between bags are simple: treelike!
-     ⇒ PARITY in P

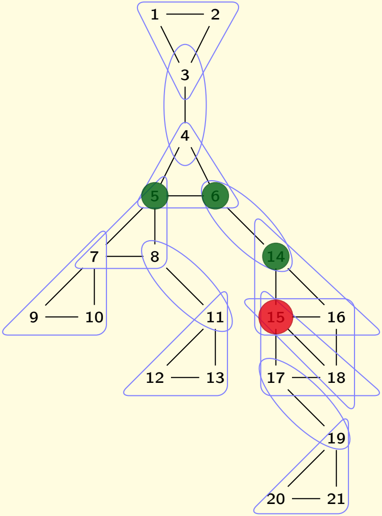


Dynamic Programming on Tree Decomposition

- small bags ⇒ arbitrary computations on leaf bags
- use: connections between bags are simple: treelike!
- ⇒ PARITY in P



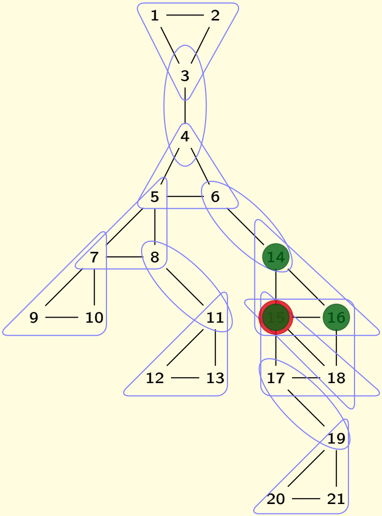
Game Theoretical Characterisation



Game rules:

- *k* Cops, one Robber
- Robber runs along cop free paths
- Cops fly
- Cops want to capture Robber

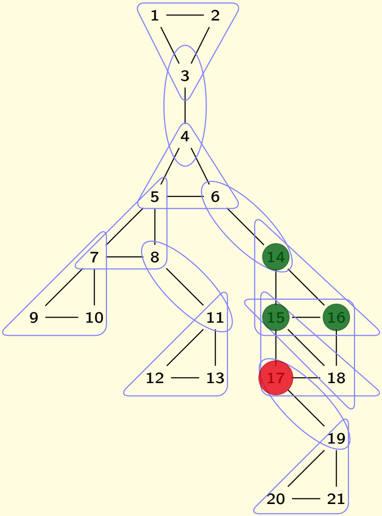
Game Theoretical Characterisation



Game rules:

- *k* Cops, one Robber
- Robber runs along cop free paths
- Cops fly
- Cops want to capture Robber

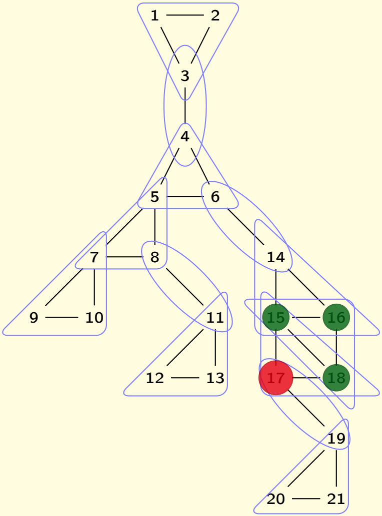
Game Theoretical Characterisation



Game rules:

- *k* Cops, one Robber
- Robber runs along cop free paths
- Cops fly
- Cops want to capture Robber

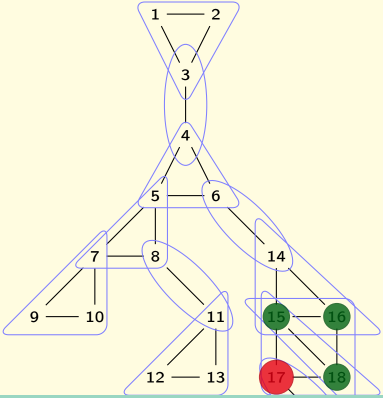
Game Theoretical Characterisation



Game rules:

- k Cops, one Robber
- Robber runs along cop free paths
- Cops fly
- Cops want to capture Robber

Game Theoretical Characterisation

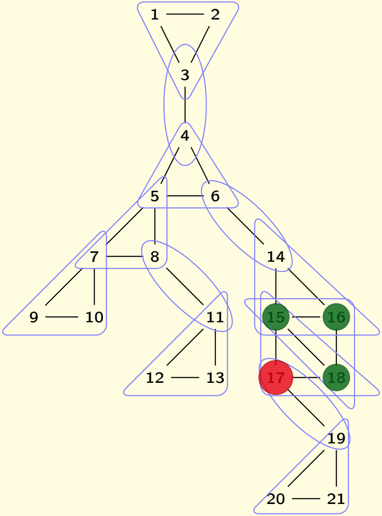


Game rules:

- k Cops, one Robber
- Robber runs along cop free paths
- Cops fly
- Cops want to capture Robber

Tree-width = minimal number cops (monotonously) capturing Robber − 1

Game Theoretical Characterisation



Game rules:

- k Cops, one Robber
- Robber runs along cop free paths
- Cops fly
- Cops want to capture Robber

Path-width:
the same,
but robber invisible.

On Directed Graphs

- DAG-width game (Berwanger et al. 2006, Obdržálek 2006)
  - like before, but Robber runs along directed paths + monotonicity
  - if DAG-width bounded (+ directed path-width): PARITY in P $\oplus$
- Kelly-width (Kreutzer, Hunter 2007)
  - like DAG-width, but Robber is inert and invisible + monotonicity
  - if Kelly-width bounded: PARITY in P $\oplus$
- directed tree-width (Johnson et al. 2001)
  - like DAG-width, but Robber doesn't leave her SCC
  - if directed tree-width bounded: PARITY in P ?
- entanglement (Grädel, Berwanger 2005)
  - like DAG-width, but Robber slow and cops restricted
  - if entanglement bounded: PARITY in P $\oplus$

$$\begin{array}{ccccc} & & & \text{dpw} & \\ & & \rotatebox{-45}{$\leq$} & & \\ \text{Kelly} \sim (?) & \text{DAG} & \leq & \text{ent} & \geq \\ & & \rotatebox{45}{$\leq$} & & \\ & & & \text{tw} & \end{array}$$

On Directed Graphs

- DAG-width game (Berwanger et al. 2006, Obdržálek 2006)
  - like before, but Robber runs along directed paths + monotonicity
  - if DAG-width bounded (+ directed path-width): PARITY in P $\oplus$
- Kelly-width (Kreutzer, Hunter 2007)
  - like DAG-width, but Robber is inert and invisible + monotonicity
  - if Kelly-width bounded: PARITY in P $\oplus$
- directed tree-width (Johnson et al. 2001)
  - like DAG-width, but Robber doesn't leave her SCC
  - if directed tree-width bounded: PARITY in P ?
- entanglement (Grädel, Berwanger 2005)
  - like DAG-width, but Robber slow and cops restricted
  - if entanglement bounded: PARITY in P $\oplus$

$$\begin{array}{cccccc} & & & \text{dpw} & & \\ & & \leq & & & \\ \text{Kelly} \sim (?) & \text{DAG} & \leq & \text{ent} & \Big) & \geq \\ & & \leq & & & \\ & & & \text{tw} & & \end{array}$$

On Directed Graphs

- DAG-width game (Berwanger et al. 2006, Obdržálek 2006)
  - like before, but Robber runs along directed paths + monotonicity
  - if DAG-width bounded (+ directed path-width): PARITY in P ⊕
- Kelly-width (Kreutzer, Hunter 2007)
  - like DAG-width, but Robber is inert and invisible + monotonicity
  - if Kelly-width bounded: PARITY in P ⊕
- directed tree-width (Johnson et al. 2001)
  - like DAG-width, but Robber doesn't leave her SCC
  - if directed tree-width bounded: PARITY in P ?
- entanglement (Grädel, Berwanger 2005)
  - like DAG-width, but Robber slow and cops restricted
  - if entanglement bounded: PARITY in P ⊕

$$\begin{array}{ccccc} & & & \text{dpw} & \\ & & \rotatebox{45}{$\leq$} & & \\ \text{Kelly} \sim (?) & \text{DAG} & \leq & \text{ent} & \Big) \; \rotatebox{90}{$\leq$} \\ & & \rotatebox{-45}{$\leq$} & & \\ & & & \text{tw} & \end{array}$$

On Directed Graphs

- DAG-width game (Berwanger et al. 2006, Obdržálek 2006)
  - like before, but Robber runs along directed paths + monotonicity
  - if DAG-width bounded (+ directed path-width): PARITY in P ⊕
- Kelly-width (Kreutzer, Hunter 2007)
  - like DAG-width, but Robber is inert and invisible + monotonicity
  - if Kelly-width bounded: PARITY in P ⊕
- directed tree-width (Johnson et al. 2001)
  - like DAG-width, but Robber doesn't leave her SCC
  - if directed tree-width bounded: PARITY in P ?
- entanglement (Grädel, Berwanger 2005)
  - like DAG-width, but Robber slow and cops restricted
  - if entanglement bounded: PARITY in P ⊕

$$\begin{array}{ccccc} & & & \text{dpw} & \\ & & \leq & & \\ \text{Kelly} \sim (?) & \text{DAG} & \leq & \text{ent} & \geq \\ & & \leq & & \\ & & & \text{tw} & \end{array}$$

Unbounded Imperfect Information

Measures Grow Exponentially

Theorem

Exists G^{imperf} simple, but G^{perf} complex (w.r.t. all our measures).

- very large information sets
- G^{perf} contains a huge undirected grid (grids are complex!)

Theorem

Reachability: EXPTIME-hard even if entanglement ≤ 2 and directed path-width ≤ 3. (Refine original proof for hardness by Reif.)

Theorem

Reachability: PSPACE-*hard even on DAGs.*

$\Rightarrow$ need to bound size of information sets

Bounded Imperfect Information

General Procedure

- show measure(G^{imperf}) $\leq k$, $|$information sets$| \leq r$

  $\Rightarrow$ measure(G^{perf}) $\leq f(k, r)$
- then for appropriate measures:

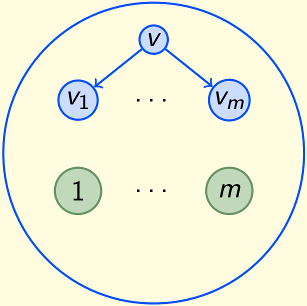
  if measure(G^{imperf}) $\leq k$ and $|$information sets$| \leq r$

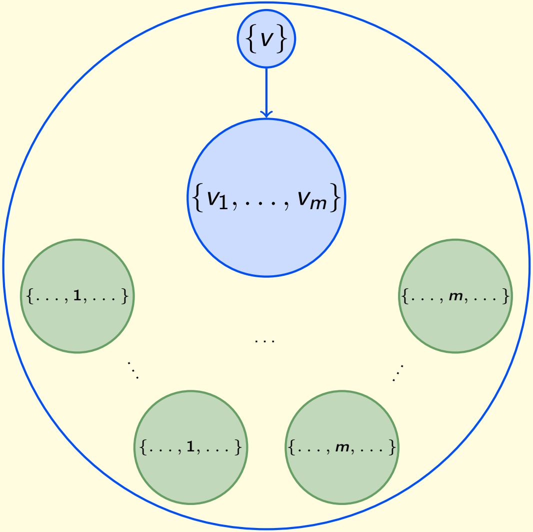
  then PARITY in P

How Measures Behave

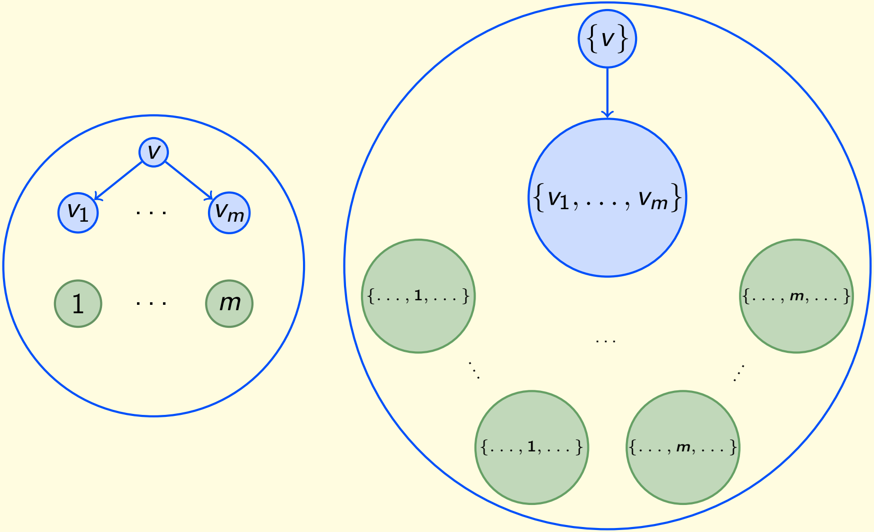
- tree-width: $\ominus$ $\mathrm{tw}(G^{imperf}) = 2$, but $\mathrm{tw}(G^{perf})$ unbounded
- entanglement: $\ominus$ $\mathrm{ent}(G^{imperf}) = 2$, but $\mathrm{ent}(G^{perf})$ unbounded
- *non-monotone* DAG-width: $\oplus$ $f(k, r) = k \cdot r \cdot 2^{r-1}$
  - not an appropriate measure
  - if every information set is an SCC: $\oplus$, $f(k, r) = k \cdot r^2 \cdot 2^{r-1}$
  - if every $|\text{information set}| \leq 2$: $\oplus$, $f(k, r)$ bounded
- DAG-width: ? good hope that $\oplus$
- directed path-width: $\oplus$, $f(k, r) = k \cdot 2^{r-1}$
- Kelly-width: ? (idea doesn't work: inert robber)
- directed tree-width: ? (doesn't work: robber cannot leave her SCC)

General Proof Idea for Boundedness of Measures

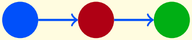
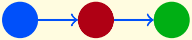




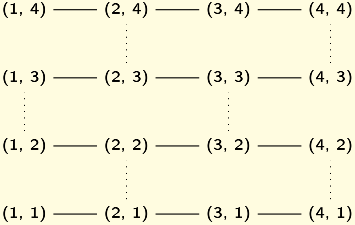
General Proof Idea for Boundedness of Measures

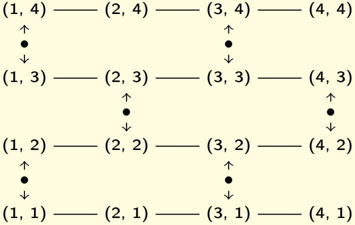


Lemma

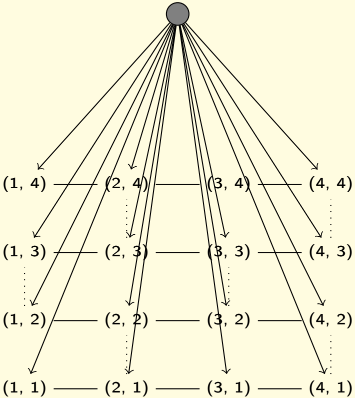
Path  *in* G^{imperf} $\Leftrightarrow$ *path*  *in* G^{perf}.

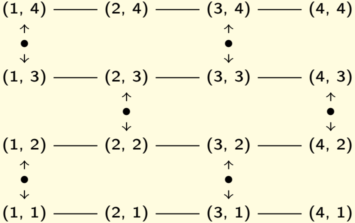
Tree-width Grows Unbounded



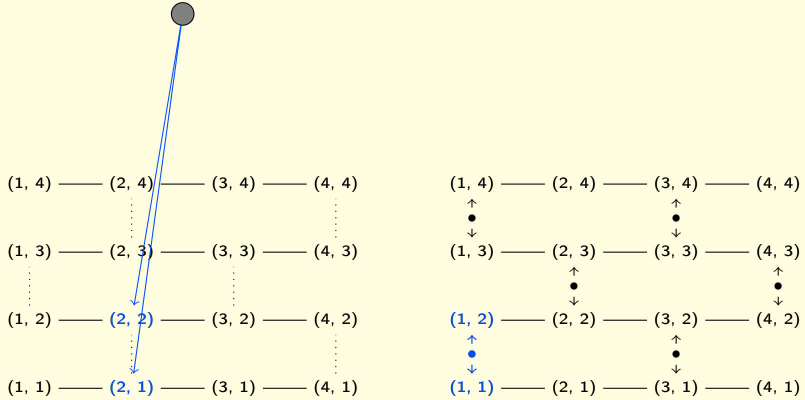


Tree-width Grows Unbounded





Tree-width Grows Unbounded



Future Work + Work in Progress

- (monotone) DAG-width of G^{imperf} bounded
  $\Downarrow$
  (monotone) DAG-width of G^{perf} bounded
- clique-width (measure for regularity of a graph)