Parity Games with Imperfect Information on Graphs of Bounded Complexity

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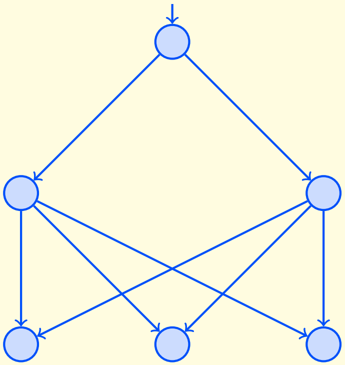
Imperfect Information



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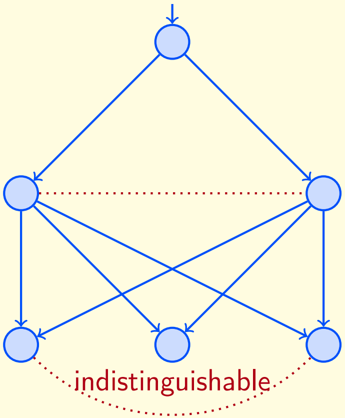

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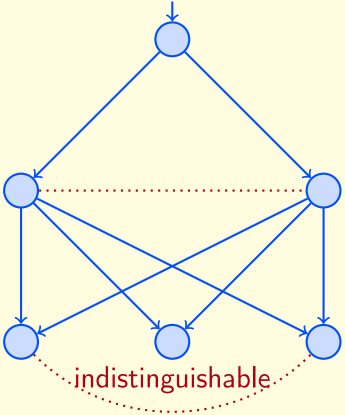
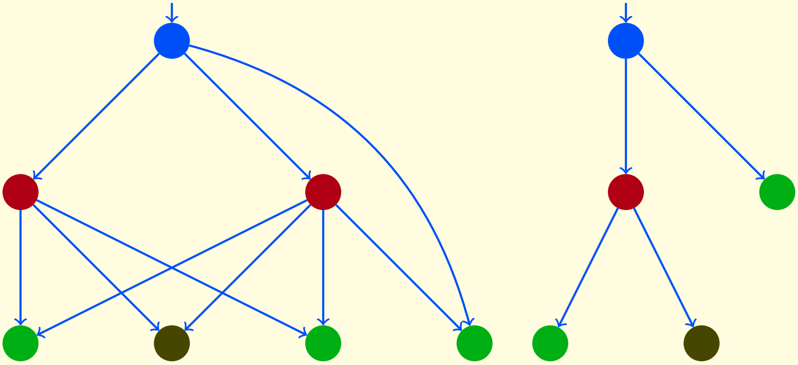


- Parity games: $(V, V_0, v_0, (E_a)_{a \in A}, \Omega)$
  - E_a deterministic
- Indistinguishable vertices build an information set.
- Technical subtleties:
  - Edge labels may be indistinguishable.
  - Edge labels compatible with information sets.
  - Colours compatible with information sets.
  - Start vertex distinguishable from all.

Powerset Construction



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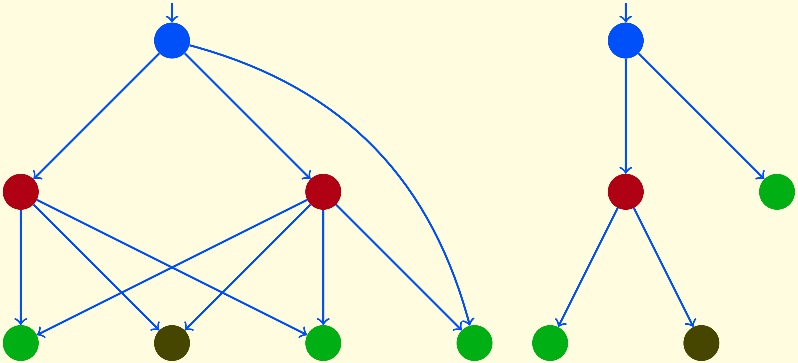
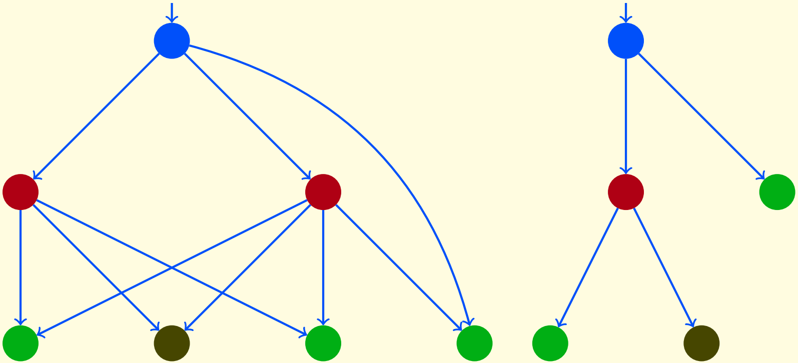


Theorem (Reif)

Player 0 wins from v in G^{imp} $\Leftrightarrow$ Player 0 wins from $\{v\}$ in G^{perf}.

Powerset Construction



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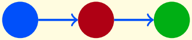
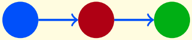
Path  *in G^{imp} $\Leftrightarrow$ path*  *in G^{perf}.*

Complexity Measures

Motivation:

- solve parity games in P despite imperfect information,
- but:
  - not known whether PARITY is in P
  - the powerset graph can be exponentially bigger

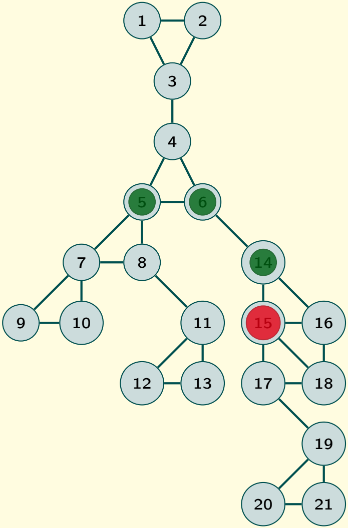
Hope: on simple graphs PARITY in P (like without imperfect information)

$\Rightarrow$ Need to measure complexity of a graph.

Measures:

- tree-width (undirected graphs) + path-width $\oplus$
- directed tree-width ?
- DAG-width + directed path-width $\oplus$
- Kelly-width $\oplus$
- entanglement $\oplus$
- . . . (not considered here)

Tree-width: Game Theoretial Definition



Game rules:

- k Cops, one Robber
- Robber runs along cop free paths
- Cops fly
- Cops want to capture Robber

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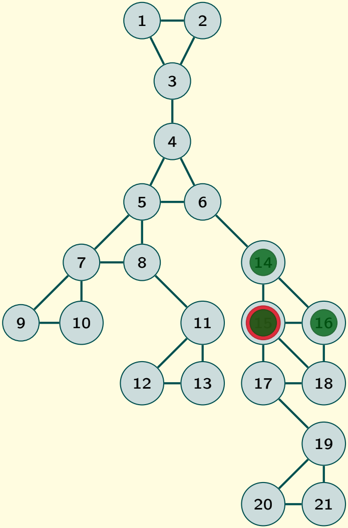

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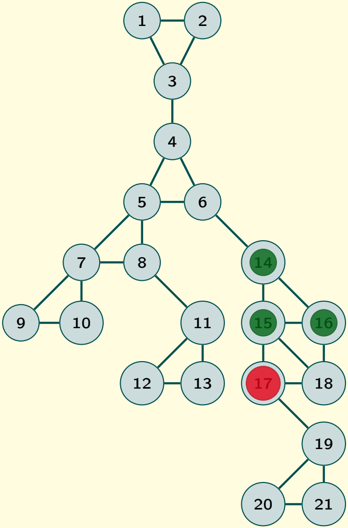

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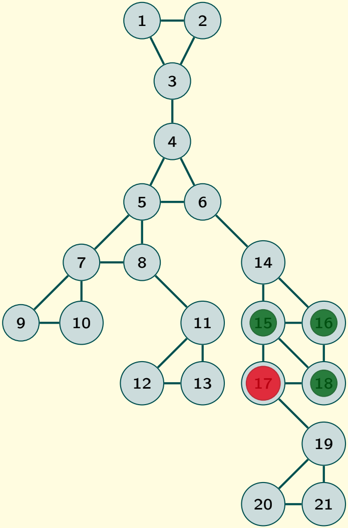

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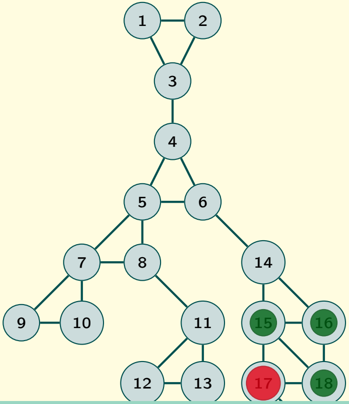

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Tree-width = minimal number cops monotonously capturing Robber − 1

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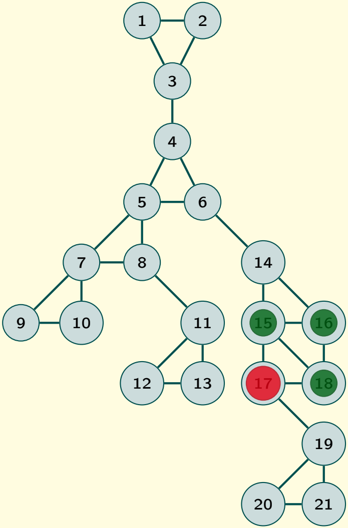

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Path-width:
the same,
but robber invisible.

On Directed Graphs

- DAG-width game (Berwanger et al. 2010)
  - like before, but Robber runs along directed paths + monotonicity
  - if DAG-width bounded (+ directed path-width): PARITY in P $\oplus$
- Kelly-width (Kreutzer, Hunter 2007)
  - like DAG-width, but Robber is inert and invisible + monotonicity
  - if Kelly-width bounded: PARITY in P $\oplus$
- directed tree-width (Johnson et al. 2001)
  - like DAG-width, but Robber doesn't leave her SCC
  - if directed tree-width bounded: PARITY in P ?
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Unbounded Imperfect Information

Measures Grow Exponentially

Theorem
Exists G^{imp} simple, but G^{perf} complex (w.r.t. all our measures).

- very large information sets
- G^{perf} contains a huge undirected grid (grids are complex!)

Theorem
Reachability: EXPTIME-hard even if entanglement ≤ 2 and directed path-width ≤ 3. (Refine original proof for hardness by Reif.)

Theorem
Reachability: PSPACE*-hard even on DAGs.*

$\Rightarrow$ need to bound size of information sets

Bounded Imperfect Information

General Procedure

- show measure(G^{imp}) $\leq k$, $|$information sets$| \leq r$

  $\Rightarrow$ measure(G^{perf}) $\leq f(k, r)$

- then for appropriate measures:

  if measure(G^{imp}) $\leq k$ and $|$information sets$| \leq r$

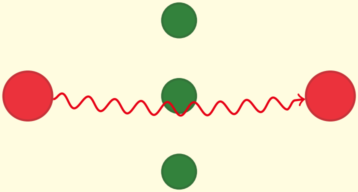
  then PARITY in P

How Measures Behave

- tree-width: $\ominus$ $\mathrm{tw}(G^{imp}) = 2$, but $\mathrm{tw}(G^{perf})$ unbounded
  - still: $\oplus$ $\mathrm{tw}(G^{imp})$ bounded $\Rightarrow$ $\mathsf{DAG}(G^{perf})$ bounded
- entanglement: $\ominus$ $\mathrm{ent}(G^{imp}) = 2$, but $\mathrm{ent}(G^{perf})$ unbounded
- directed path-width: $\oplus$, $f(k,r) = k \cdot 2^{r-1}$
- *non-monotone* DAG-width: $\oplus$ $f(k,r) = k \cdot r \cdot 2^{r-1}$
  - not an appropriate measure
  - if every information set is an SCC: $\oplus$, $f(k,r) = k \cdot r^2 \cdot 2^{r-1}$
  - if every $|$information set$|$ ≤ 2: $\oplus$, $f(k,r)$ bounded
- DAG-width: $\oplus$ a newer result: $f(k,r) = 3kr \cdot 2^{r-1}$
- Kelly-width: ? (idea doesn't work: inert robber)
- directed tree-width: ? (doesn't work: robber cannot leave her SCC)

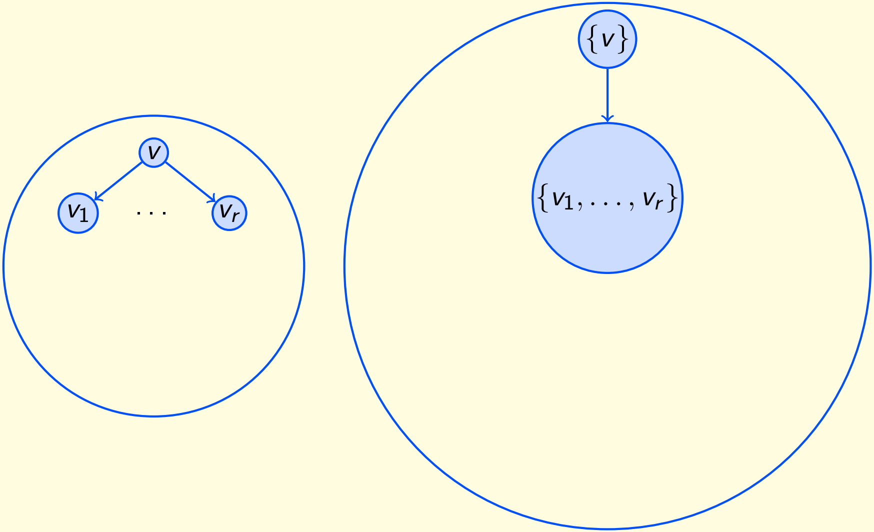
New DAG-Game: Multiple Robbers

- Cops must capture *r* Robbers.
- Robbers can jump:



- Monotonicity:
  no robber can access a vertex that was inaccessible to all robbers

General Proof Idea for Boundedness of Measures



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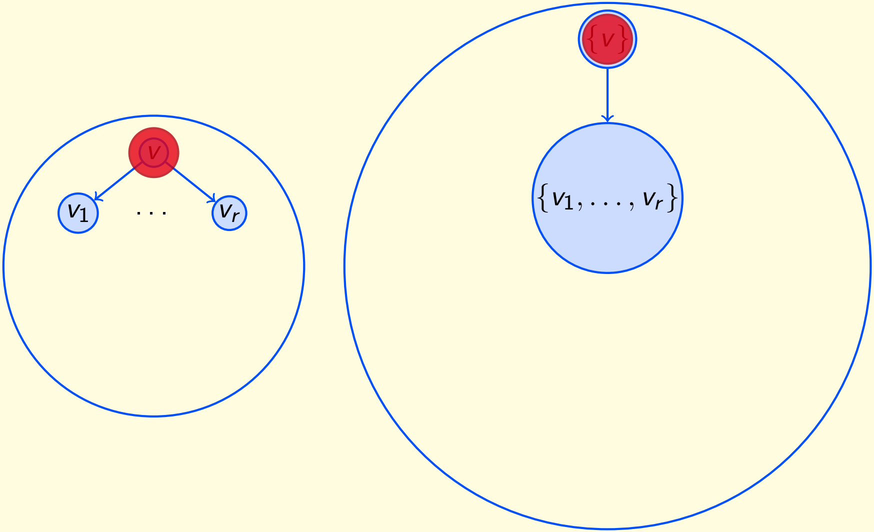

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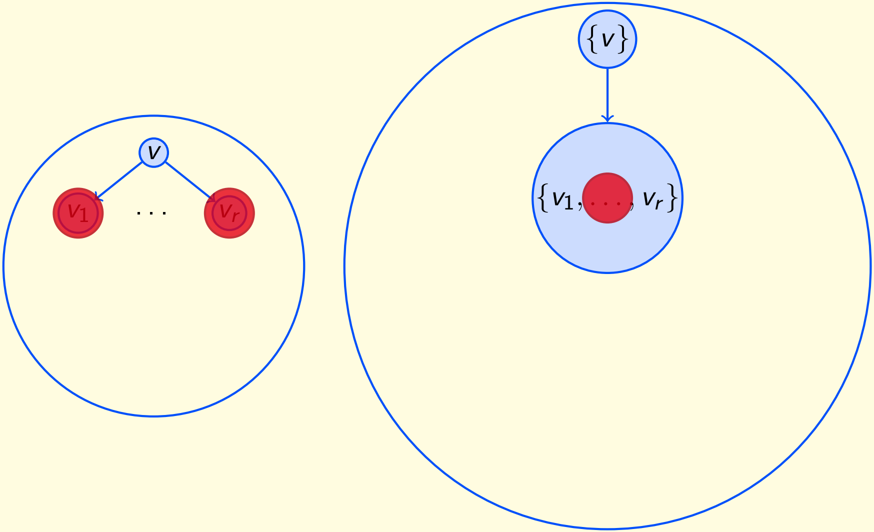

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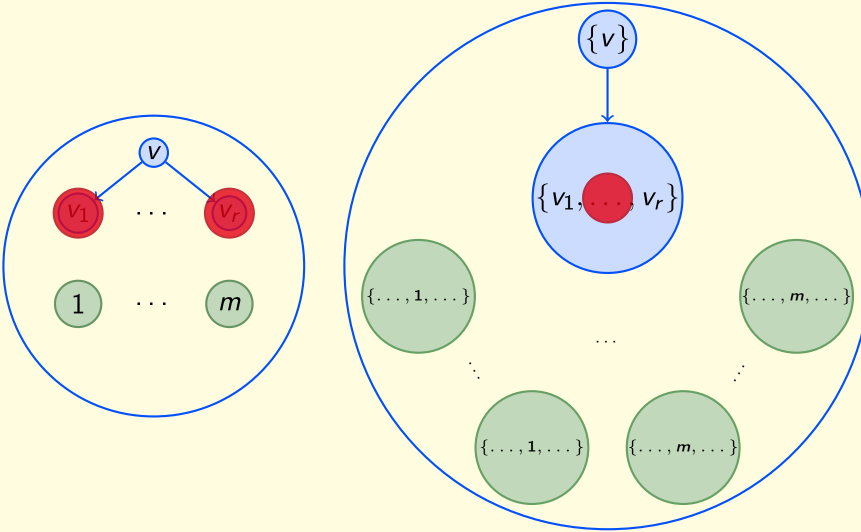

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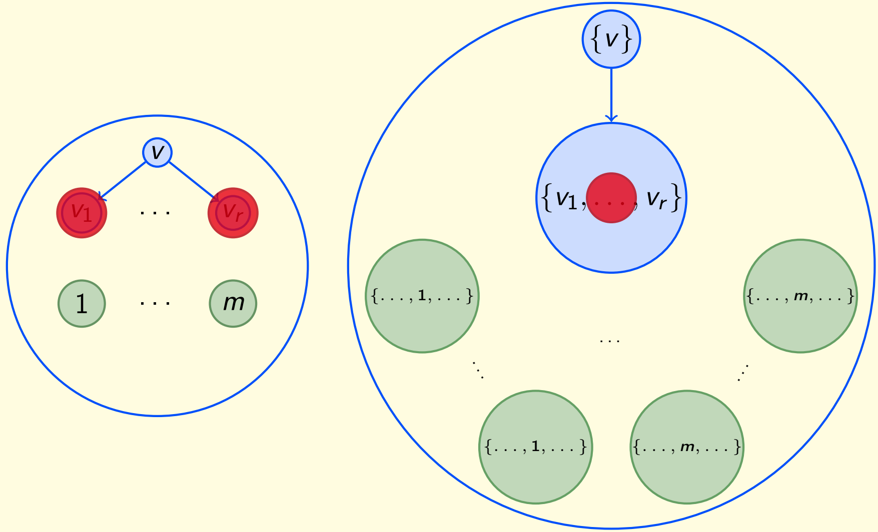

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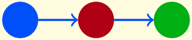
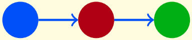
Path  *in* $G^{imp} \Leftrightarrow$ *path*  *in* G^{perf}.

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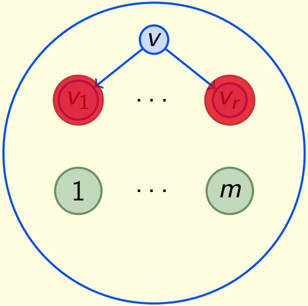
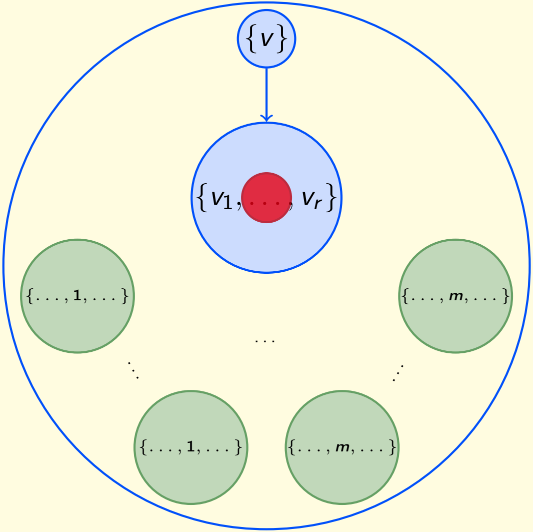
nm-DAG-width:
$f(k,r) = k \cdot r \cdot 2^{r-1}$





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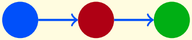
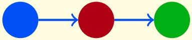
Path  *in* G^{imp} $\Leftrightarrow$ *path*  *in* G^{perf}.

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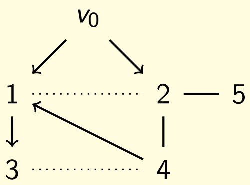

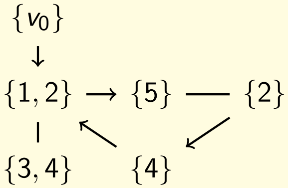

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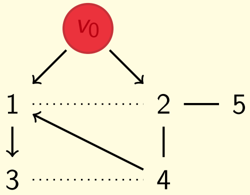

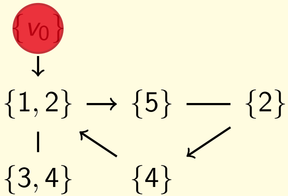

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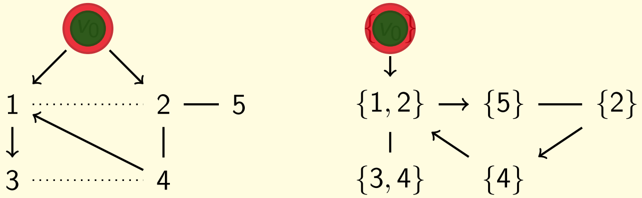

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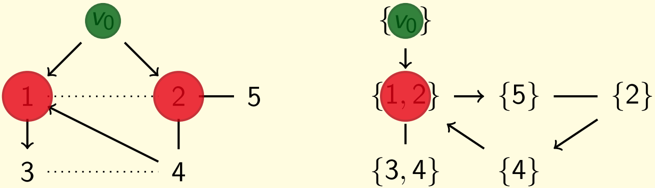

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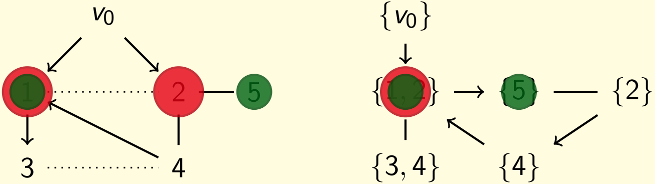

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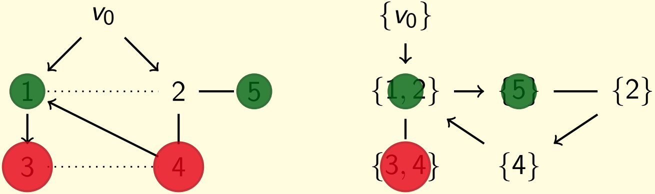

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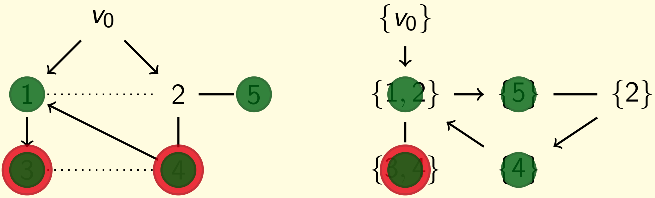

Preserving Monotonicity

For a moment, assume that:
k cops win Tree-width Game

$\Leftrightarrow$ forget directions of edges and win

Stronger assumption: robber has more paths to run, but monotonicity costs are zero!

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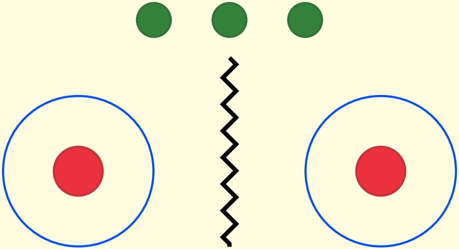
If k cops monotonously win against one robber then $k \cdot r$ cops monotonously win against r robbers.



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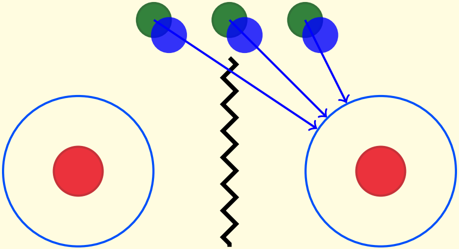

Summing up for Tree-width

Theorem

If $|\text{information set}| \leq r$ and tree-width of $G \leq k$ then PARITY is efficiently solvable on G.

What doesn't work for DAG-width?

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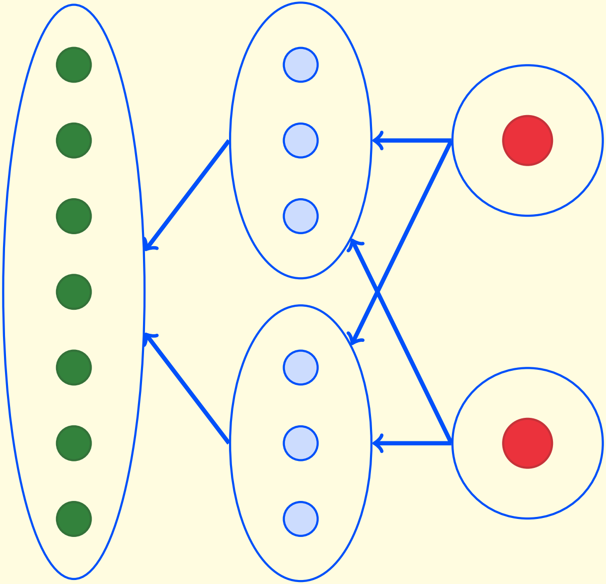

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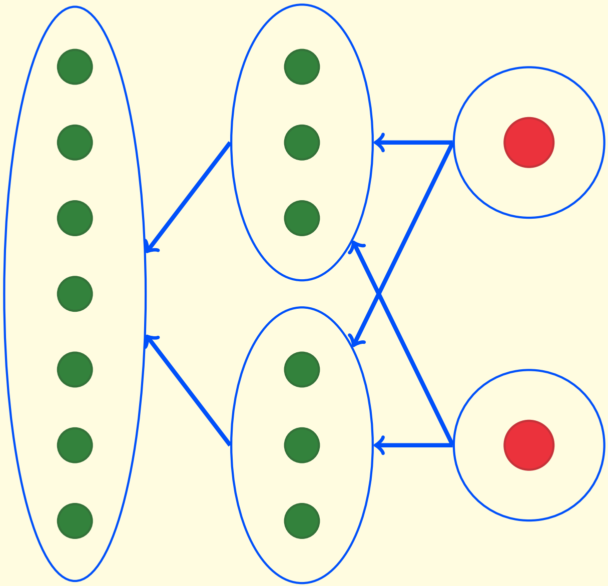

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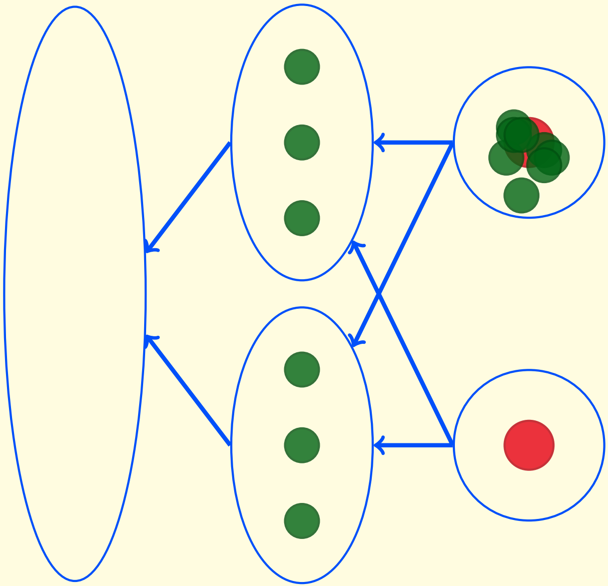

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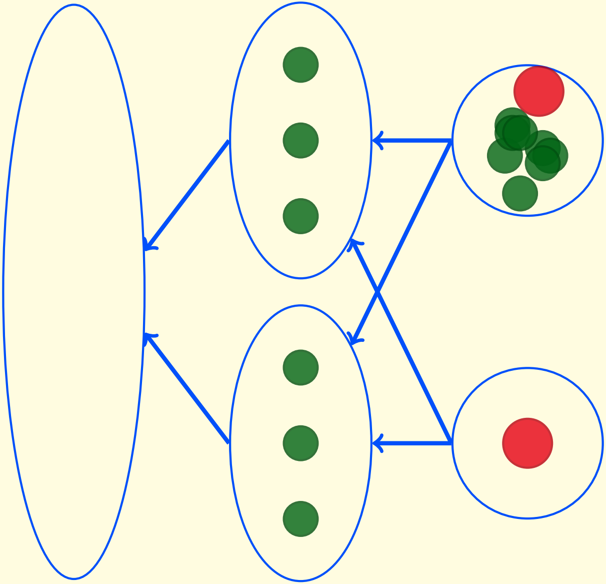
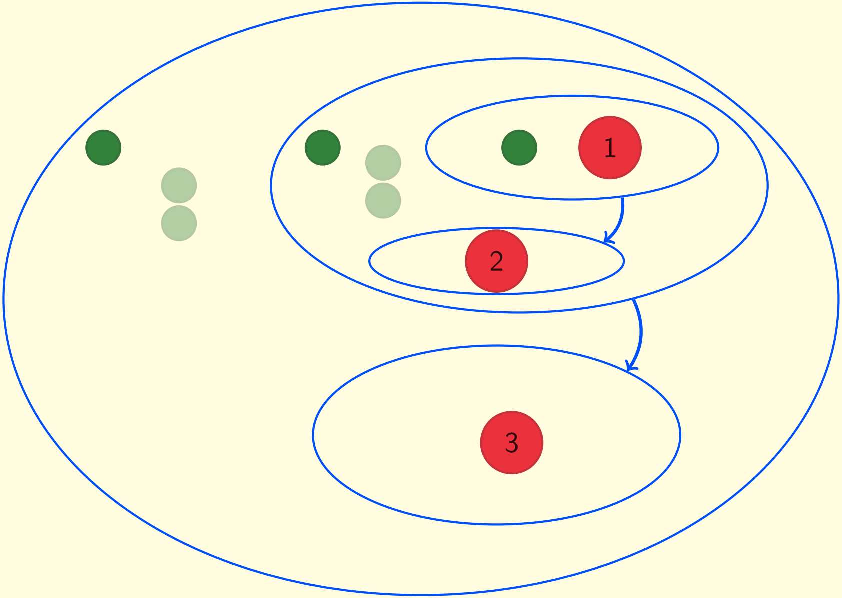


Solving Monotonicity Problem for r Robbers



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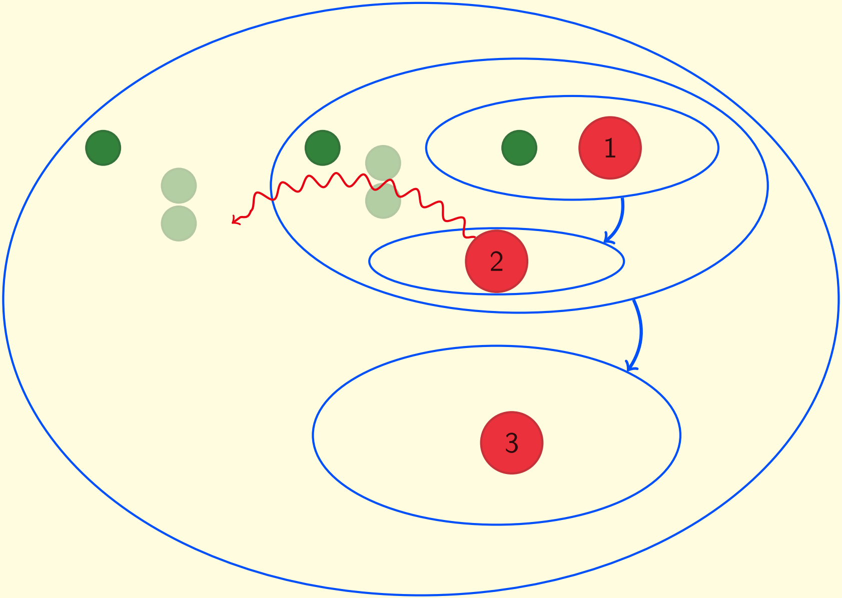

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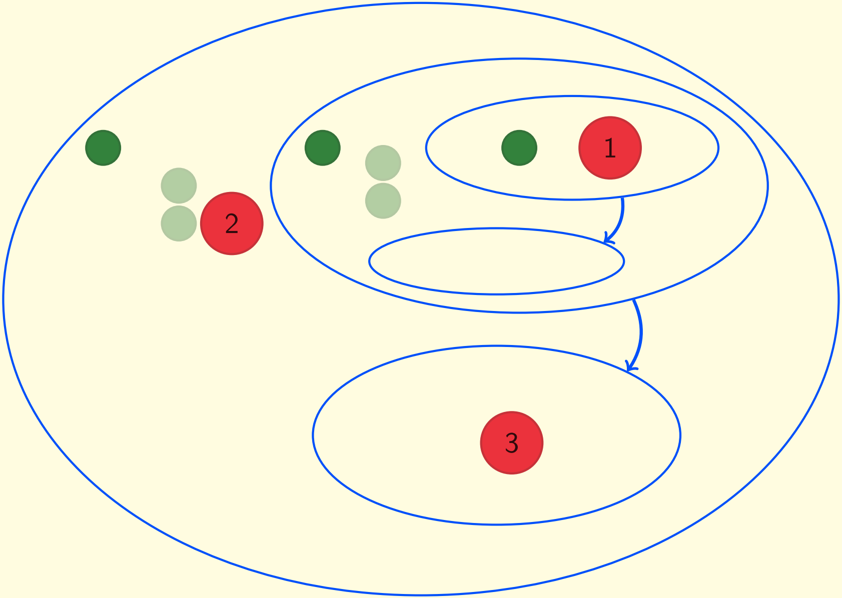

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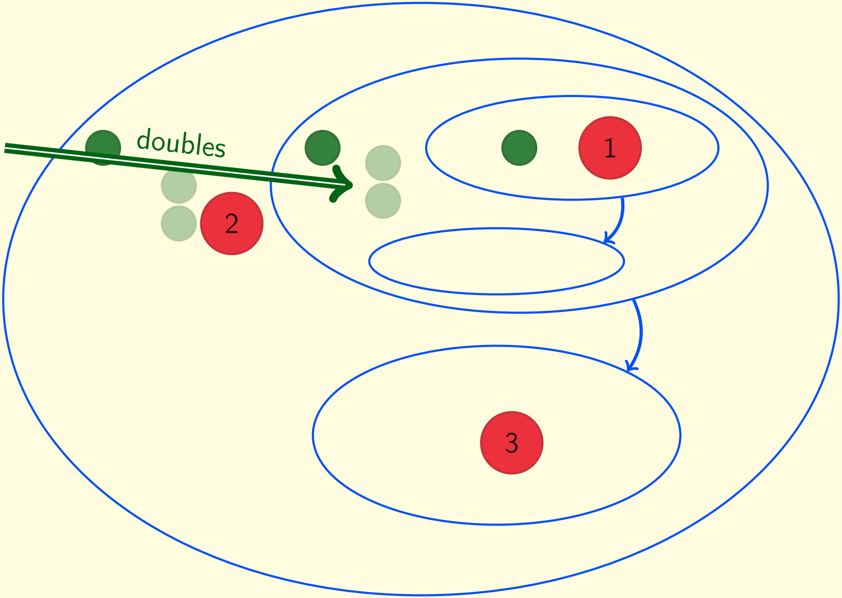

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