# Parity Games with Imperfect Information and Complexity Measures

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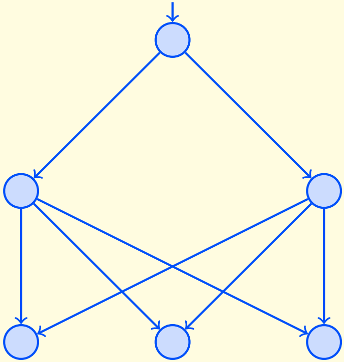
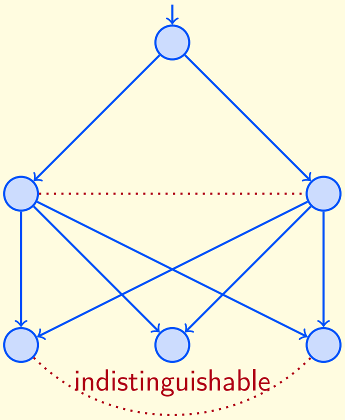
June 2011

# Parity Games with Imperfect Information



- Parity games: $(V, V_0, v, (E_a)_{a \in A}, \Omega)$
- Indistinguishable vertices build an information set.

# Parity Games with Imperfect Information

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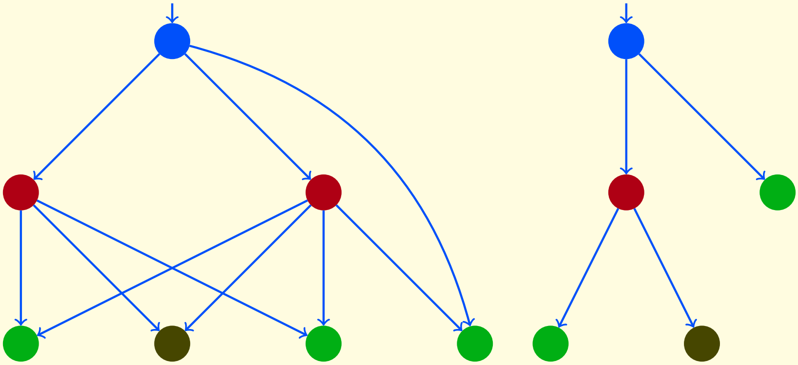
# Powerset Construction (Reif 1984)

Information tracking with perfect recall:



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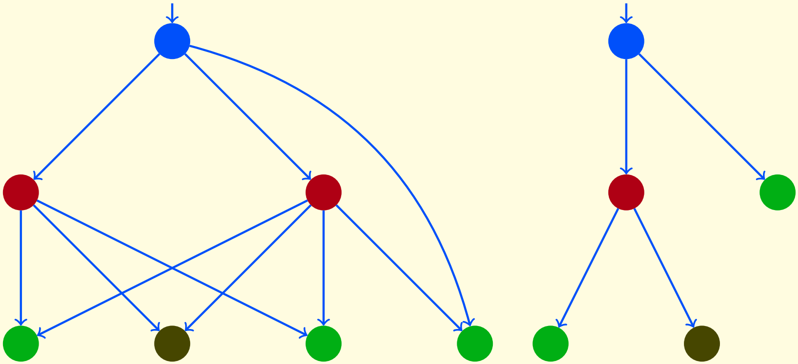


### Theorem (Reif)

*Player 0 wins from $v$ in $G^{imp}$ $\Leftrightarrow$ Player 0 wins from $\{v\}$ in $G^{perf}$.*

# Complexity Measures

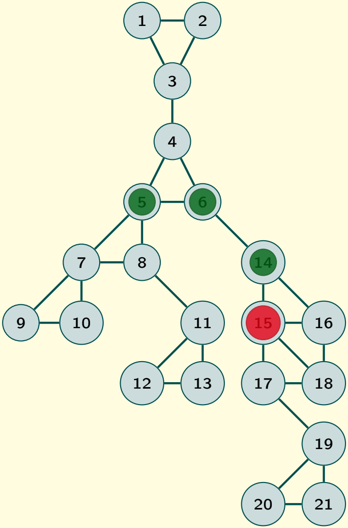
Motivation:

- solve parity games in P despite imperfect information,
- but:
  - not known whether PARITY is in P
  - the powerset graph can be exponentially larger

Hope: on simple graphs PARITY in P (like without imperfect information)

$\Rightarrow$ Need to measure complexity of a graph.

# Tree-width: Game Theoretical Definition




Game rules:

- $k$ Cops, one Robber
- Robber runs along cop free paths
- Cops fly
- Cops want to capture Robber

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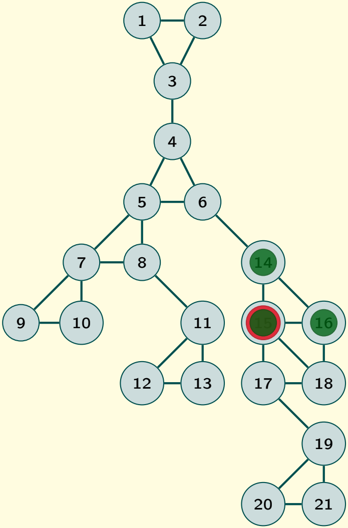


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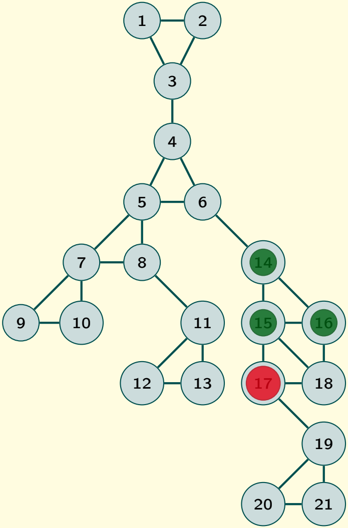


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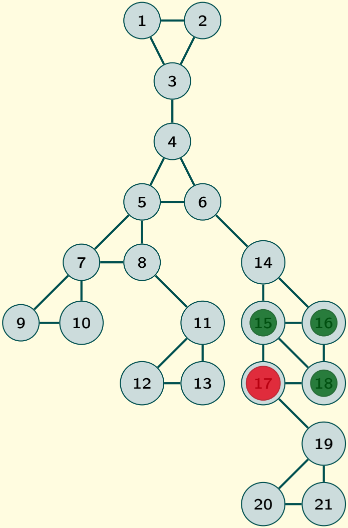


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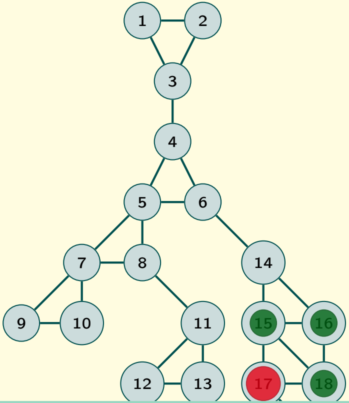


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Tree-width = minimal number cops monotonously capturing Robber − 1

# On Directed Graphs

- **tree-width** (forget directions of edges) ⊕ (Obdržálek 2003)
- directed tree-width ?
- **DAG-width** ⊕ (Berwanger et al. '06; Obdržálek '06)
- Kelly-width ⊕ (Kreutzer, Hunter 2008)
- entanglement ⊕ (Berwanger, Grädel 2005)
- ...
- no game characterisation:
  - clique-width
  - rank-width
  - bi-rank-width
- ...

# DAG Game

- DAG-width game (Berwanger et al. 2006; Obdržálek 2006)
  - like before, but Robber runs along directed paths (+ monotonicity)
  - monotonicity costs: positive (Kreutzer, Ordyniak, 2008)
  - DAG-width bounded ⇒ PARITY in P ⊕

Interesting questions (asked a year ago):

  - Monotonicity bounded? (most interesting) — Weak DAG games, Kaiser, Puchala, R.
  - Many more offhanded cops? — Yes, Kaiser, Puchala, R.
  - Many more cops if many Robbers? — Yes, Puchala, R., in this talk.
  - Computing DAG-width in NP? — Many people say yes, but nobody knows.

# Unbounded Imperfect Information

*(General Case)*

*everything is bad...*

# Measures Grow Exponentially

### Theorem (Puchala, R.)

*Exists $G^{imp}$ of small complexity, but $G^{perf}$ of exponential complexity (w.r.t. all our measures).*

- very large information sets

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- further restrictions needed
- natural approach: bound size of information sets

# Bounded Imperfect Information

# General Procedure

- show for appropriate $\oplus$ measures:

**Lemma**

$\mathrm{measure}(G^{imp}) \leq k$, $|information\ sets| \leq r$
$\Rightarrow \mathrm{measure}(G^{perf}) \leq f(k, r)$

- then
  - if $\mathrm{measure}(G^{imp}) \leq k$ and $|\text{information sets}| \leq r$
  - ($\Rightarrow |G^{perf}|$ polynomial in $|G^{imp}|$)

  - $\Rightarrow$ PARITY in P

# How Measures Behave (Puchala, R.)

- tree-width: $\ominus$ $treewidth(G^{imp}) = 2$, but $treewidth(G^{perf})$ unbounded
  - **still:** $\oplus$ $treewidth(G^{imp})$ bounded $\Rightarrow$ DAG($G^{perf}$) bounded
- entanglement: $\ominus$ $entanglement(G^{imp}) = 2$, but $entanglement(G^{perf})$ unbounded
- Kelly-width: ?
- directed tree-width: ? (idea doesn't work)
- DAG-width:
  - $\oplus$ a new result: $f(k, r) = kr \cdot 2^{r-1}$

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- DAG-width:
  - $\oplus$ a new result: $f(k,r) = kr \cdot 2^{r-1}$
    use DAG game with $r$ robbers

Thank you for your attention!