

The Complexity of Nash Equilibria in Infinite Multiplayer Games

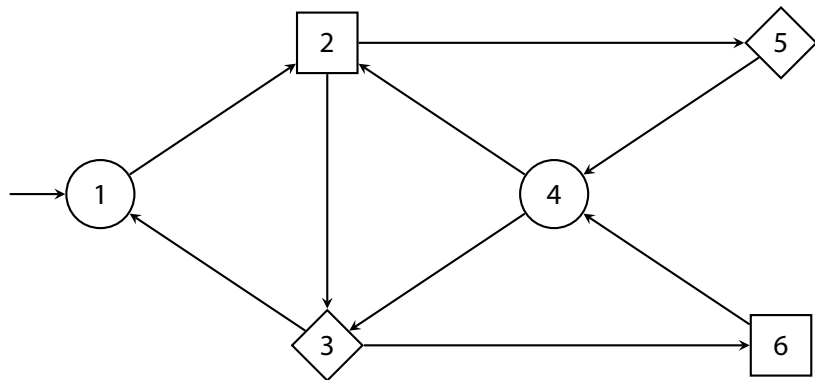
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Infinite Games

Let's play!



Play: $\pi = 1, 2, 3, 6, 4, 2, 5, \dots$

Note: No probabilistic vertices!

Winning conditions

Question: What is the payoff of a play?

Specified by a **winning condition** for each player:

- ▶ **Büchi condition:** Given a set F of vertices, defines the set of all plays π that hit F *infinitely often*.
- ▶ **Co-Büchi condition:** Given a set F of vertices, defines the set of all plays π that hit F only *finitely often*.
- ▶ **Parity condition:** Given a *priority function* $\Omega : V \rightarrow \mathbb{N}$, defines the set of all plays π such that the least priority occurring infinitely often is even.

Player receives payoff 1 if her winning condition is satisfied, otherwise 0.

But we are not so much interested in the winner of a certain play, but in the **strategic behaviour** that can occur.

The Classical Case

Two-player Zero-sum Games: Games with two players where the winning conditions are complements of each other.

(Pure) Determinacy: A two-player zero-sum game is **determined (in pure strategies)** if one of the two players has a (pure) winning strategy.

Theorem (Martin 1975)

Any two-player zero-sum game with a Borel winning condition is determined in pure strategies.

Solving these games corresponds to deciding which player wins.

Theorem (Emerson & Jutla 1991)

The problem of deciding the winner in a parity game is $NP \cap co-NP$.

Question: What is the solution of a multiplayer game?

Nash Equilibria

Definition: A (mixed) strategy profile is a **Nash equilibrium** if no player can gain from unilaterally switching to a different strategy.

Pure Nash Equilibrium: If you get payoff 0 in the equilibrium play, you get payoff 0 with any other strategy.

Motivation: In a Nash equilibrium, no player has an incentive to switch strategies.

Question: Do Nash equilibria always exist?

Theorem (Chatterjee & al. 2004)

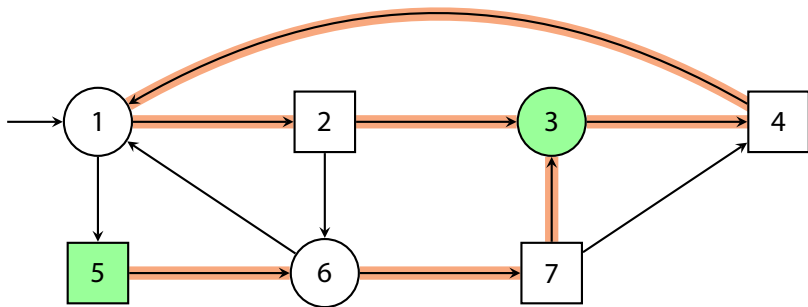
Any infinite multiplayer game with Borel winning conditions has a Nash equilibrium in pure strategies.

But there may be many Nash equilibria (with different payoffs)...

Example

Winning condition for player 0 (round vertices): Visit 3 infinitely often.

Winning condition for player 1 (square vertices): Visit 5 infinitely often.



Fact: For every $\bar{x} \in [0, 1]^2$ there is a (mixed) Nash equilibrium with payoff \bar{x} !

The Problem NE

Two computational problems:

- ▶ Compute *some* Nash equilibrium.
- ▶ Compute one that meets certain requirements on the payoff.

We focus on the second (more involved) task.

The problem NE: Given a game \mathcal{G} , two binary payoff thresholds $\bar{x}, \bar{y} \in \{0, 1\}^k$, decide whether the game has a (mixed) Nash equilibrium with payoff $\geq \bar{x}$ and $\leq \bar{y}$.

Thresholds determine which players may win and which players shall lose.

Recall: The classical parity game problem of deciding the winner is in $\text{NP} \cap \text{co-NP}$.

Question: What is the complexity of NE?

Related Work

The computational complexity of finding Nash equilibria has gained much interest in recent years:

- ▶ The problem of finding *some* Nash equilibrium in a finite two-player game is PPAD-complete (Daskalakis & al. 2006, Chen & Deng 2006).
- ▶ Finding a Nash equilibrium *with certain properties* in a finite two-player game is NP-hard (Gilboa & Zemel 1989, Conitzer & Sandholm 2003).

Problem: Models are not comparable.

Our results:

- ▶ NE is in NP for games with parity winning conditions.
- ▶ NE is NP-hard for games with co-Büchi winning conditions.
- ▶ NE is in P for games with Büchi winning conditions (only).

Characterising Nash Equilibria

Necessary condition for having a Nash equilibrium with payoff $\geq \bar{x}$ and $\leq \bar{y}$:

- ▶ There must be a play with such a payoff.
- ▶ This play must avoid the *winning region* of each player i with $y_i = 0$.

Key observation: This condition is also sufficient!

Proof idea:

- ▶ Let the players try to build up the given play.
- ▶ If some player deviates, let all the other players switch to a joint counter-strategy (threat/trigger strategy).

Remark: Proof reveals that pure finite-state strategies suffice (fails if thresholds are not binary!).

Deciding NE

Theorem

NE is in NP for games with parity winning conditions.

Here is an NP algorithm for NE:

- ▶ Guess a (binary) payoff $\bar{x} \leq \bar{z} \leq \bar{y}$.
- ▶ “Compute” the winning region W_i of each player $i \in \Pi$.
- ▶ Check whether there exists a play with payoff \bar{z} starting in v and avoiding W_i for each player i with $z_i = 0$.

The last step amounts to checking the existence of a winning play in a one-player Streett game: can be done in polynomial time (Emerson-Lei)!

Remark: If the number of players is bounded, we get the same complexity as for the parity game problem.

Theorem

NE is NP-hard for games with co-Büchi winning conditions.

The proof is by a reduction from SAT.

Given a formula φ in CNF over variables X_1, \dots, X_k , we construct (in polynomial time) a co-Büchi game \mathcal{G}_φ with players $0, 1, \dots, k$ such that:

φ is satisfiable

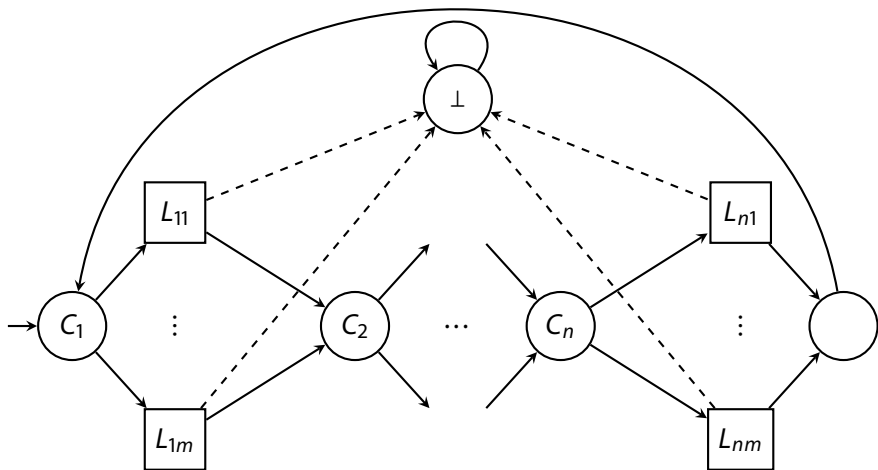
$\Leftrightarrow \mathcal{G}_\varphi$ has a Nash equilibrium where player 0 wins.

$\Leftrightarrow \mathcal{G}_\varphi$ has a Nash equilibrium with payoff $\geq (1, 0, \dots, 0)$
and $\leq (1, 1, \dots, 1)$.

The Construction

The game \mathcal{G}_φ :

Edges to \perp only from negated variables!



What you might want to remember from this talk:

Finding Nash equilibria in infinite games is hard!

However:

- ▶ If the number of players is bounded, NE retains the complexity of the parity game problem.
- ▶ For games with Büchi winning conditions only, NE is in P.