

The Complexity of Nash Equilibria in Simple Stochastic Multiplayer Games

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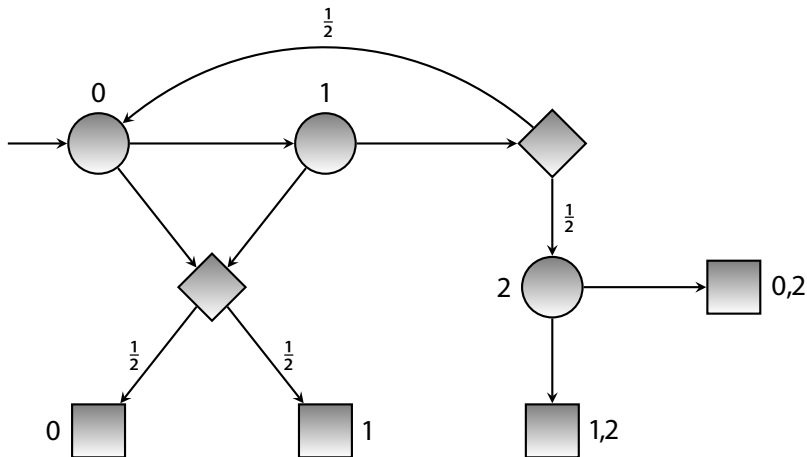


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Simple Stochastic Games

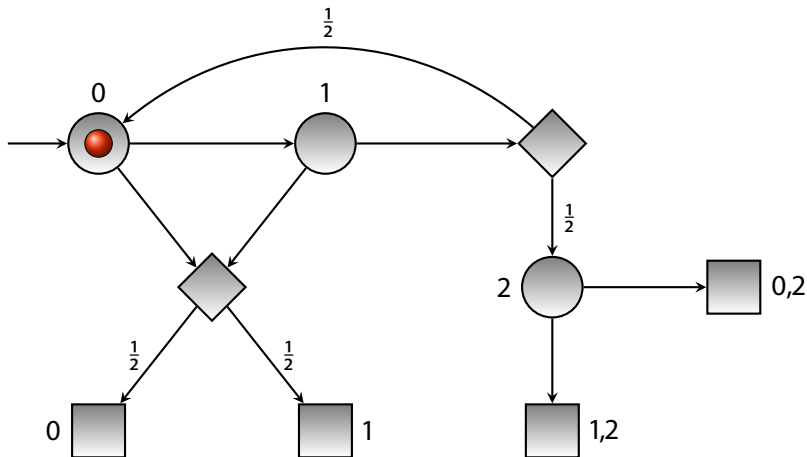
What is a simple stochastic multiplayer game (SSMG)? [Let's play!](#)



Winners are decided at terminal vertices. Infinite plays are *bad*.

Simple Stochastic Games

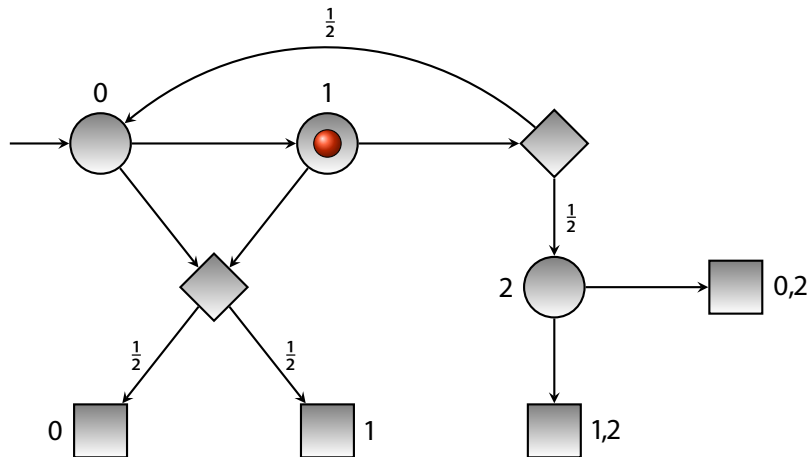
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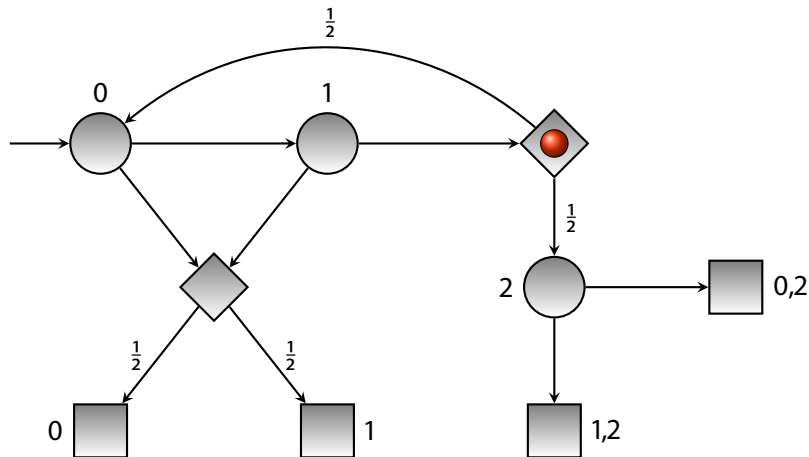
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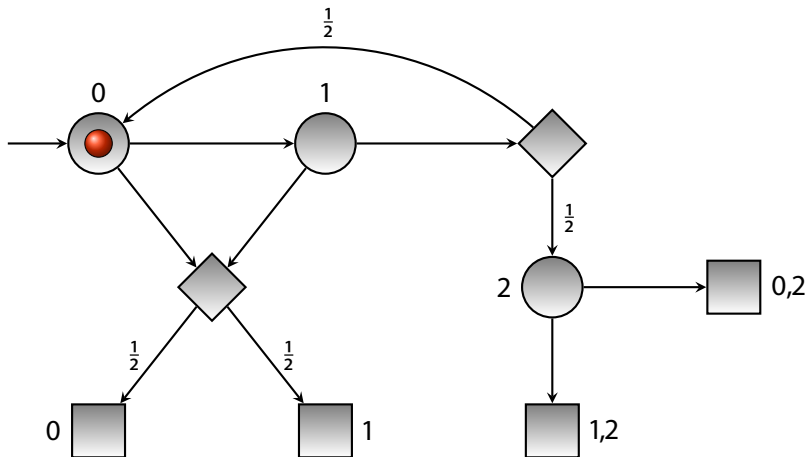
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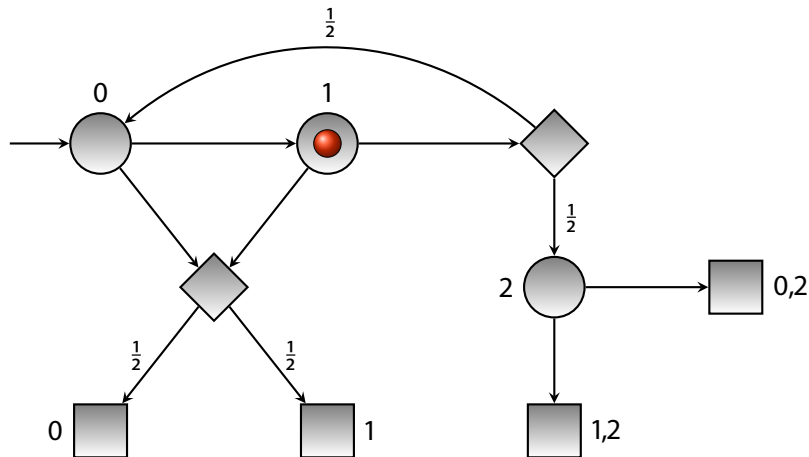
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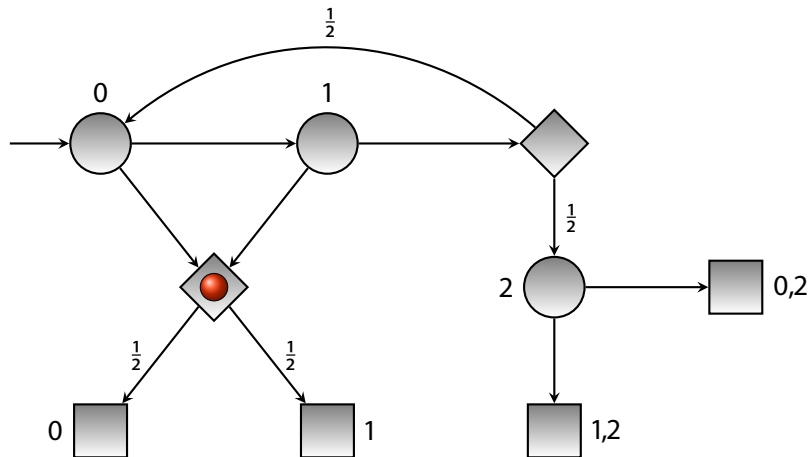
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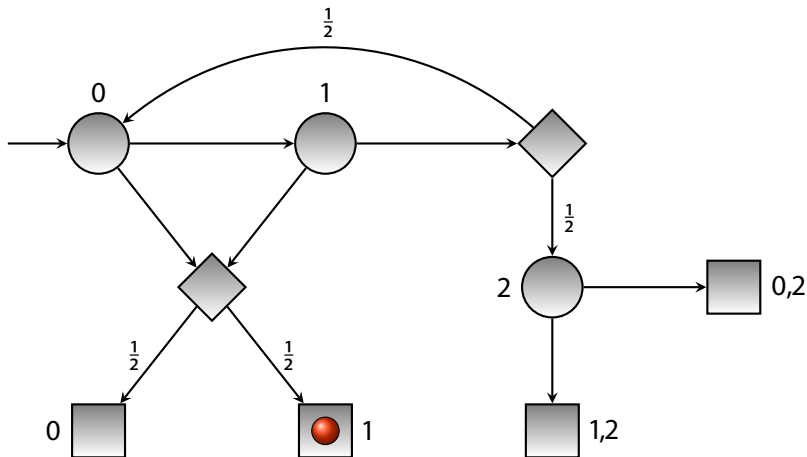
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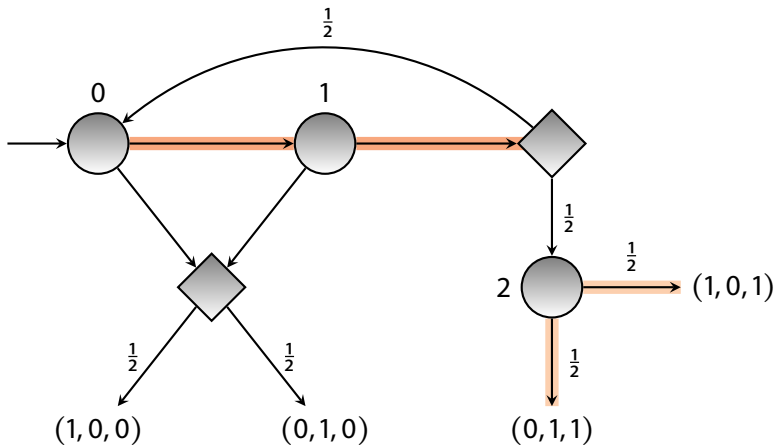
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Strategies and Probabilities

A strategy profile induces a probability distribution on sets of plays.



Payoff of this strategy profile: $(\frac{1}{2}, \frac{1}{2}, 1)$

Two-Player Zero-Sum Games

Classical setting: Games played by two players 0 and 1 with opposing objectives ([Two-Player Zero-sum Games](#)).

Theorem (Liggett-Lippman, 1969; Condon, 1992)

Simple stochastic two-player zero-sum games are **determined**, i.e.

$$\sup_{\sigma} \inf_{\tau} \text{pay}_0(\sigma, \tau) = \inf_{\tau} \sup_{\sigma} \text{pay}_0(\sigma, \tau) =: \text{val}(\mathcal{G})$$

Both players have optimal pure stationary strategies.

Corollary

The decision problem, given a simple stochastic zero-sum game \mathcal{G} and $p \in [0, 1]$, decide whether $\text{val}(\mathcal{G}) \geq p$ is in $\text{NP} \cap \text{co-NP}$.

Open Problem: Can the value be computed in polynomial time?

Nash Equilibria

Definition: A strategy profile is a **Nash equilibrium** if no player can gain from unilaterally switching to a different strategy.

Question: Do Nash equilibria always exist?

Theorem (Chatterjee & al., 2004)

Any SSMG has a Nash equilibrium in pure finite-state strategies.

Next Question: Can we compute one?

Proposition

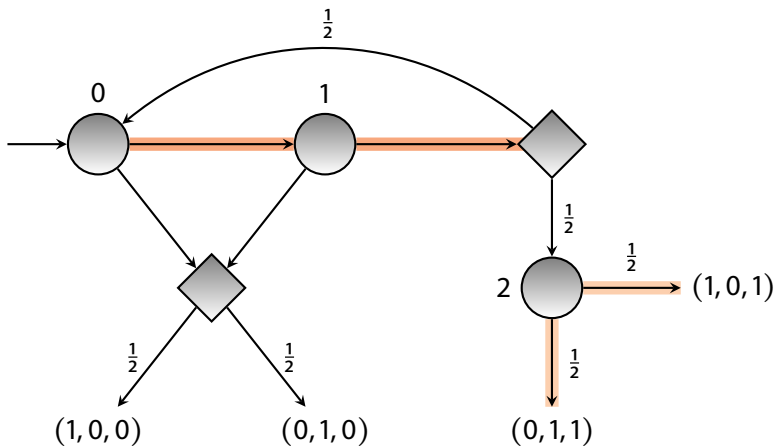
The problem of computing a (pure finite-state) Nash equilibrium of an SSMG is in FNP.

Open Problem: Can one compute a Nash equilibrium in polynomial time?

But there may be many Nash equilibria (with different payoffs)...

Example

Nash Equilibrium where Player 2 wins (with probability 1):



Observation: Mixed strategies are more powerful...

The Problem NE

Goal: Compute a Nash equilibrium that meets certain requirements on the payoff.

The problem NE: Given an SSMG \mathcal{G} , two payoff thresholds $\bar{x}, \bar{y} \in [0, 1]^k$, decide whether the game has a Nash equilibrium with payoff $\geq \bar{x}$ and $\leq \bar{y}$.

Special case: Given an SSMG \mathcal{G} , decide whether the game has a Nash equilibrium where Player 0 wins almost surely.

Variants of the Problem:

- ▶ Arbitrary strategies
- ▶ Pure strategies
- ▶ Stationary strategies
- ▶ Pure stationary strategies

The Complexity of NE (and the special case):

	Pure	Mixed
Memoryful	Undecidable	NP-hard
Stationary	NP-complete	PSPACE *

* NP-hard and at least as hard as SQRT-SUM

Pure stationary Nash equilibria

Deciding the existence of a pure stationary NE with payoff $\geq \bar{x}$ and $\leq \bar{y}$:

- ▶ Guess a pure stationary strategy profile $\bar{\sigma}$.
- ▶ For each player i :
 1. Compute the payoff r_i of $\bar{\sigma}$ for player i .
 2. Compute the maximal payoff z_i player i can achieve by herself.
 3. Check whether $x_i \leq r_i \leq y_i$ and $z_i \leq r_i$.

1. and 2. are doable in polynomial time (via linear programming).

Theorem

NE for pure stationary strategies is in NP.

Remark: NP-hardness holds even for games with only two players and if arbitrary stationary strategies are allowed.

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Stationary Nash Equilibria

Deciding the existence of a stationary NE with payoff $\geq \bar{x}$ and $\leq \bar{y}$:

- ▶ Guess the support S of a stationary strategy profile $\bar{\sigma}$.
- ▶ For each player i , compute the set R_i of vertices from where the set of winning terminal vertices is reachable when playing $\bar{\sigma}$.
- ▶ Evaluate an existential first-order sentence ψ (which is polynomial-time computable from $\mathcal{G}, \bar{x}, \bar{y}, S$ and $(R_i)_{i \in \text{Players}}$) over $\mathfrak{R} = (\mathbb{R}, +, \cdot, 0, 1)$.

ψ states that there exists a stationary Nash equilibrium $\bar{\sigma}$ with payoff $\geq \bar{x}$ and $\leq \bar{y}$ whose support is precisely S .

Note: The existential theory of \mathfrak{R} is decidable in PSPACE.

Theorem

NE is in NPSPACE, and hence in PSPACE, for stationary strategies.

The Square-Root-Sum Problem

Square-Root-Sum Problem (SQRT-SUM): Given $d_1, \dots, d_n \in \mathbb{N}$ and $k \in \mathbb{N}$, decide whether $\sum_{i=1}^n \sqrt{d_i} \geq k$.

The precise complexity of SQRT-SUM is not known:

- ▶ Known to be in PSPACE (actually in the 4th level of the counting hierarchy).
- ▶ No non-trivial lower bounds known.

Open Problem (since 1970s): Does SQRT-SUM lie inside the polynomial hierarchy? Is it in NP?

Theorem

There is a polynomial-time reduction from SQRT-SUM to NE for stationary strategies.

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Undecidability

Undecidability is shown by a reduction from the non-halting problem for two-counter machines.

Theorem

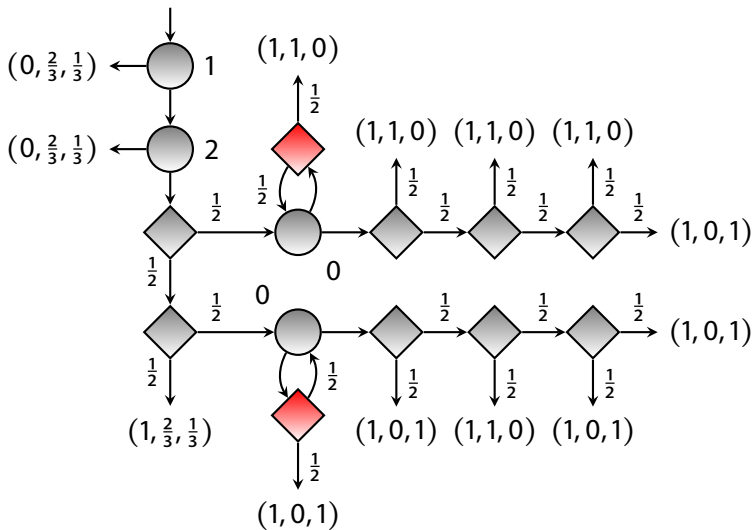
For every two-counter machine \mathcal{M} one can (algorithmically) construct a nine-player SSMG \mathcal{G} such that the computation of \mathcal{M} is infinite iff \mathcal{G} has a pure Nash equilibrium where player 0 wins almost surely.

Corollary

NE is undecidable for pure strategies.

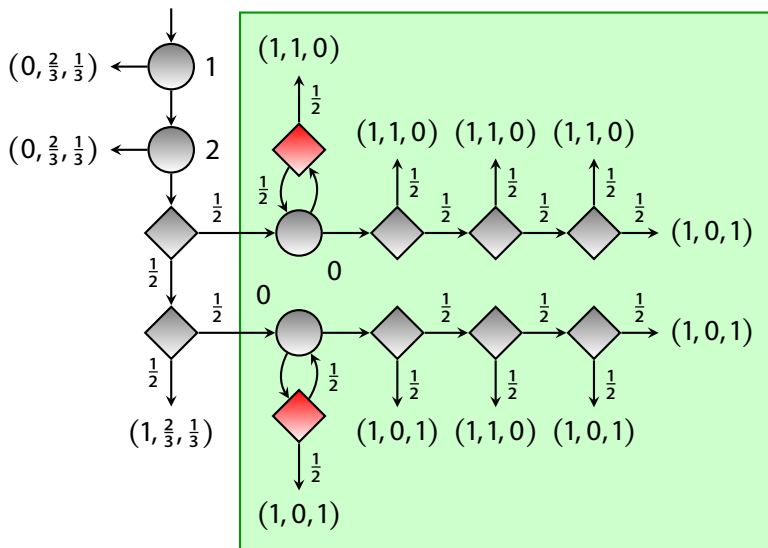
Remark: The reduction is similar to one used by Brázdil et al. (2006) for showing that stochastic games with *branching-time* winning conditions are undecidable.

Simulating a Counter



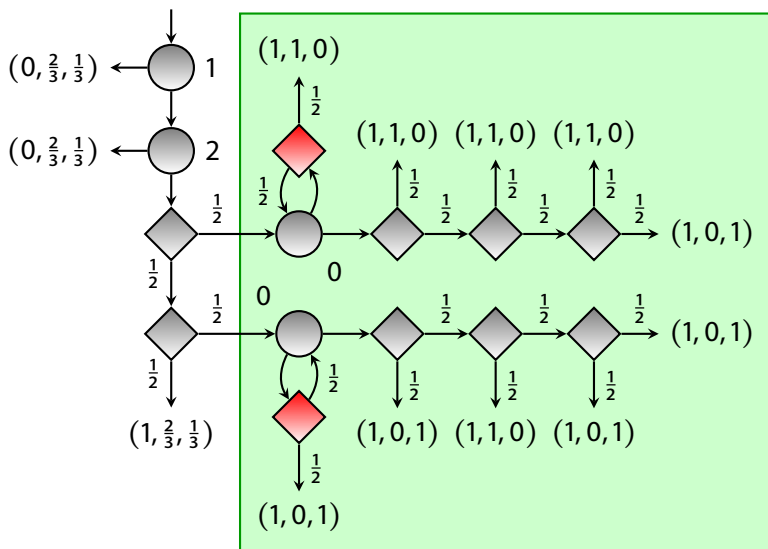
Encoding of the counter: (Maximal) number of visits to the red vertex.

Simulating a Counter



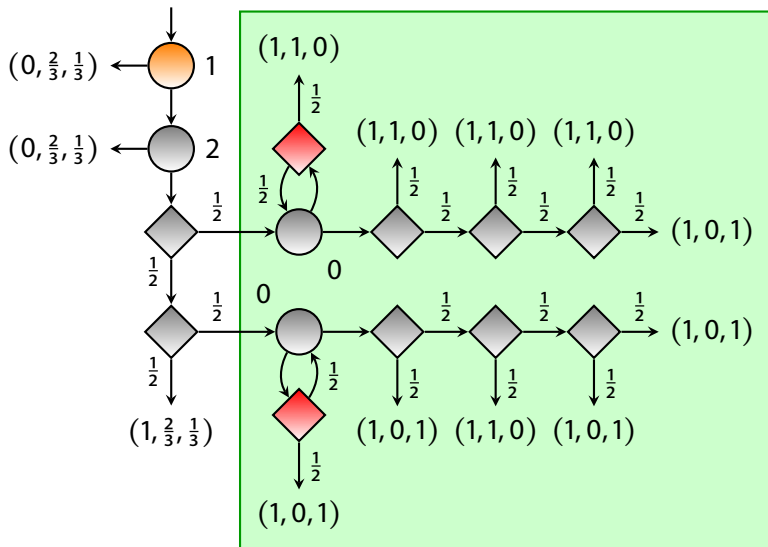
Payoff for player 1 in NE where player 0 wins almost surely: $p + \frac{1}{4} \cdot \frac{2}{3}$.

Simulating a Counter



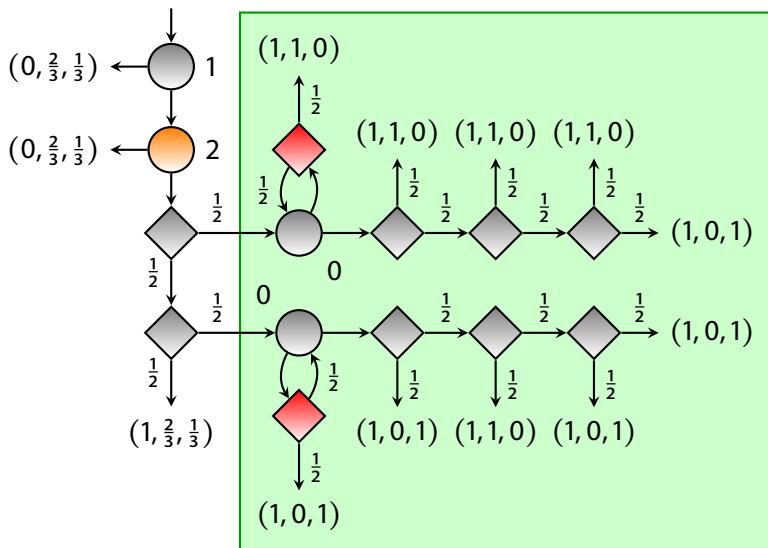
Payoff for player 1 in NE where player 0 wins almost surely: $p + \frac{1}{6}$.

Simulating a Counter



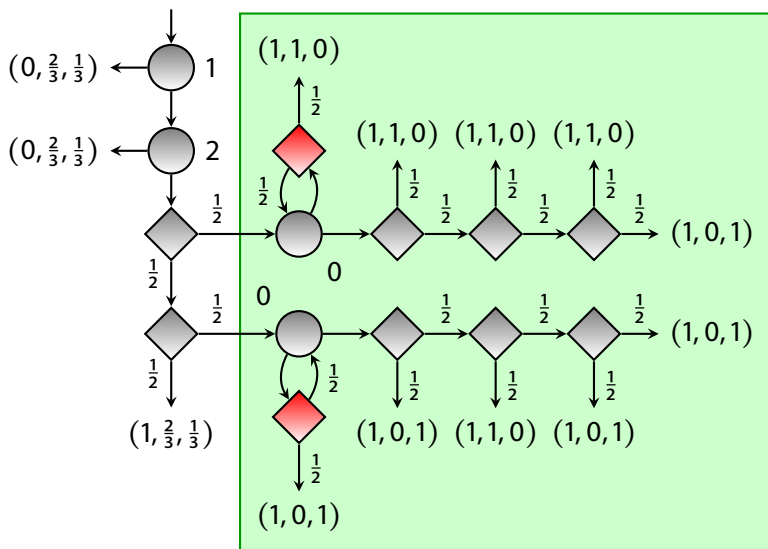
Payoff for player 1 in NE where player 0 wins almost surely: $p + \frac{1}{6} \geq \frac{2}{3}$.

Simulating a Counter



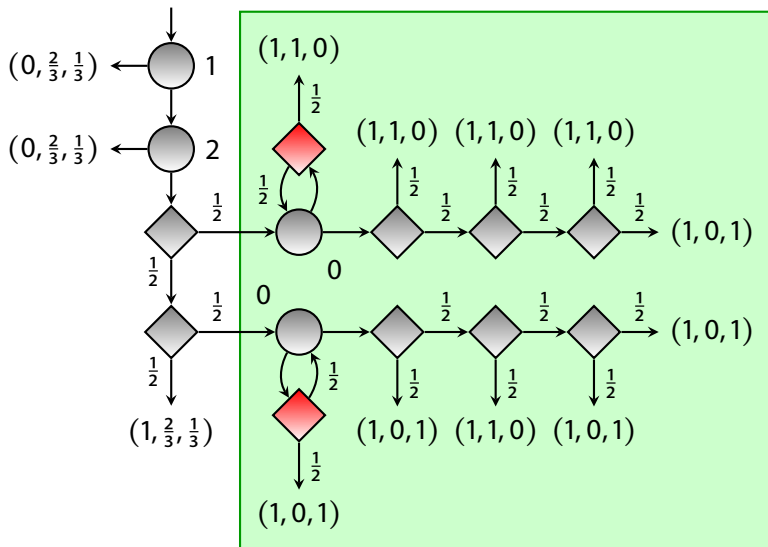
Payoff for player 1 in NE where player 0 wins almost surely: $p + \frac{1}{6} = \frac{2}{3}$.

Simulating a Counter



Necessary for NE where player 0 wins almost surely: $p = \frac{1}{2}$.

Simulating a Counter



Necessary for NE where player 0 wins almost surely: Counter maintained.

Finite-State Nash Equilibria

A consequence of the proof is:

Theorem

There exists an SSMG \mathcal{G} with a pure Nash equilibrium where player 0 wins with probability 1, but in any pure finite-state Nash equilibrium player 0 wins with probability 0.

Question: Is NE decidable when restricted to pure finite-state strategies?

Remark: NE for pure finite-state strategies is recursively enumerable.

Theorem

NE is undecidable for pure finite-state strategies.

The proof is by a reduction from the halting problem for two-counter machines (similar to the one before).

What to take home?

Finding good Nash equilibria in simple stochastic games is really hard.

Future work:

- ▶ Mixed Nash equilibria.
- ▶ Games with few players.
- ▶ Infinite-state games.